Université de Franche-Comté
Méthodes diophantiennes.

Small points on varieties of algebraic tori

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1 From torsion to small points

The former Manin-Mumford conjecture predicts that the set of torsion points of a curve of genus ≥ 2 embedded in its jacobian is finite. More generally, let $\mathbb{G}$ be a semi-abelian variety and V an algebraic subvariety of $\mathbb{G}$, defined over some algebraically closed field K. We say that V is a *torsion variety* if it is a translate of a proper connected algebraic subgroup by a torsion point of $\mathbb{G}$. We also denote by V_{tors} the set of torsion points of $\mathbb{G}$ lying on V. We have the following generalization of the Manin-Mumford conjecture.

Theorem 1.1.

i) If V is not a finite union of torsion varieties, then V_{tors} is not Zariski dense.

ii) The Zariski closure of V_{tors} is a finite union of torsion varieties.

The two assertions are clearly equivalent. Theorem 1.1 was proved by Raynaud ([24]) when $\mathbb{G}$ is an abelian variety, by Laurent ([20]) if $\mathbb{G} = \mathbb{G}_{\text{m}}^n$, and finally by Hindry ([18]) in the general situation.

We assume from now on that all varieties are algebraic and defined over $\overline{\mathbb{Q}}$. Bogomolov ([10]) gave the following generalization of the former Manin-Mumford conjecture. Let $\mathcal{C}$ be a curve of genus ≥ 2 embedded in its jacobian. Then $\mathcal{C}(\overline{\mathbb{Q}})$ is discrete for the metric induced by a Néron-Tate height. In other words, Bogomolov conjectures that the set of points of *sufficiently small* height on $\mathcal{C}$ is finite, while the former Manin-Mumford conjecture makes a similar assertion on the set of torsion points (which are precisely the points of zero height).

More generally, let $\mathbb{G}$ be a semi-abelian variety and let $\hat{h}$ be a normalized height on $\mathbb{G}(\overline{\mathbb{Q}})$. For instance, if $\mathbb{G}$ is an abelian variety we can choose for $\hat{h}$ a Neron-Tate height, and if $\mathbb{G} = \mathbb{G}_{\text{m}}^n \hookrightarrow \mathbb{P}_n$ we can choose the restriction of

the Weil height on $\mathbb{P}_n$. In particular $\hat{h}$ is a non-negative function on $\mathbb{G}$ and $\hat{h}(P) = 0$ if and only if P is a torsion point.

Let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^n$. We define the essential minimum $\hat{\mu}^{\mathsf{ess}}(V)$ as the infimum of the set of $\theta > 0$ such that

$$V(\theta) = \{P \in V \mid \hat{h}(P) \leq \theta\}$$

is Zariski dense in V. We also define V^{u} as the union of the torsion subvarieties contained in V. By the former Manin-Mumford conjecture, V^{u} is the Zariski closure of V_{tors} and thus it is the union of the *finite set* of maximal torsion subavarieties of V (see exercise 1.2). In particular $V^* = V \backslash V^{\mathrm{u}}$ is a Zariski open set. We set $\mu^*(V) := \inf_{\boldsymbol{\alpha} \in V^*} \hat{h}(\boldsymbol{\alpha})$.

Theorem 1.2. *Let V be a subvariety of $\mathbb{G}$. Then:*

i) If V is not a union of torsion varieties, then $\hat{\mu}^{\mathsf{ess}}(V) > 0$.

ii) V^* is discrete for the metric induced by $\hat{h}$, i.e. $\mu^*(V) > 0$.

It is easy to see that the two assertions are equivalent. Theorem 1.2 was proved for $\mathbb{G} = \mathbb{G}_{\mathrm{m}}^n$ by Zhang (see [28]). Bilu (see [9]) gave an independent elementary proof, relying on his equidistribution theorem for Galois conjugates of algebraic numbers with small height. In the abelian case, Ullmo (see [26]) proved Bogomolov's original formulation for curves ($\dim(V) = 1$); immediately after Zhang (see [29]) proved theorem 1.2.

The semi-abelian case was solved by David and Philippon (see [16]).

In these lessons we state some quantitative versions of theorem 1.2 for a torus $\mathbb{G} = \mathbb{G}_{\mathrm{m}}^n$ and we sketch proofs of some of these results.

Exercice 1.1. *Prove that assertions i) and ii) of theorem 1.1 are equivalent. Do the same for theorem 1.2.*

Exercice 1.2. *We say that a torsion subvariety $B \subseteq V$ is maximal (in V), if B is not strictly contained in any torsion variety contained in V. Deduce from theorem 1.1 that the set of maximal torsion subvarieties of V is finite, and that its union is the Zariski closure of V_{tors}.*

2 Conjectures and results

Let $\alpha \in \overline{\mathbb{Q}}$ and let K be a number field containing α. We denote by $\mathcal{M}_K$ the set of places of K. For $v \in K$, let K_v be the completion of K at v and let $|\cdot|_v$ be the (normalized) absolute value of the place v. Hence

$$|\alpha|_v = |\sigma\alpha|,$$

if v is an archimedean place associated to the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. If v is a non archimedean place associated with the prime ideal $\wp$ over the rational prime p, we have

$$|\alpha|_v = p^{-\lambda/e},$$

where e is the ramification index of $\wp$ and λ is the exponent of $\wp$ in the factorization of the ideal (α) in the ring of integers of K. This standard normalization agrees with the product formula

$$\prod_{v\in\mathcal{M}_K} |\alpha|_v^{[K_v:\mathbb{Q}_v]} = 1$$

which holds for any $\alpha \in K^*$. For further references we recall that for any rational place w (thus $w = \infty$ or $w =$ a prime number)

$$\sum_{v|w} [K_v : \mathbb{Q}_v] = [K : \mathbb{Q}].$$

Let $\boldsymbol{\alpha} = (\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}_n(K)$ and let K be any number field containing $\alpha_0, \ldots, \alpha_n$. We define the Weil height of $\boldsymbol{\alpha}$ by:

$$h(\boldsymbol{\alpha}) = \frac{1}{[K:\mathbb{Q}]} \sum_{v\in\mathcal{M}_K} [K_v : \mathbb{Q}_v] \log\max\{|\alpha_0|_v, \ldots, |\alpha_n|_v\}.$$

This definition does not depend on the number field K; moreover it does not depend on the projective coordinates of $\boldsymbol{\alpha}$ (use the product formula). This provides a height function $\hat{h}(x_1, \ldots, x_n) = h(1 : x_1 : \cdots : x_n)$ on $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$. The following properties hold:

i) the function $\hat{h}$ is a positive function on $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$, vanishing only on its torsion points;

ii) $\hat{h}(\boldsymbol{\alpha\beta}) \le \hat{h}(\boldsymbol{\alpha}) + \hat{h}(\boldsymbol{\beta})$. Moreover, if $\boldsymbol{\zeta}$ is a torsion point, $\hat{h}(\boldsymbol{\zeta\alpha}) = \hat{h}(\boldsymbol{\alpha})$;

iii) if $n \in \mathbb{N}$ then $\hat{h}(\boldsymbol{\alpha}^n) = n\hat{h}(\boldsymbol{\alpha})$;

iv) a subset of $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ of bounded height and bounded degree is finite.

For a proof of the statements above and for more on Weil's height, see for instance [19].

We fix the embedding $\iota\colon \mathbb{G}_{\mathrm{m}}^n \hookrightarrow \mathbb{P}_n$, $\iota(x_1,\dots,x_n) = (1 : x_1 : \cdots : x_n)$. By subvariety of $\mathbb{G}_{\mathrm{m}}^n$ we mean an algebraic subvariety V defined over some number field K. The degree of V is the degree of its Zariski closure in $\mathbb{P}_n$. We shall say that V is irreducible if its Zariski closure is geometrically irreducible. Similarly, we say that V is K-irreducible if its Zariski closure is irreducible over K.

We denote by $[l]\colon \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^n$ the *multiplication* by $[l]$, *i.e.* the morhism $\mathbf{x} \mapsto \mathbf{x}^l = (x_1^l,\dots,x_n^l)$. Thus the kernel $\mathrm{Ker}([l])$ is the set of l-torsion points. It is a subgroup $\cong (\mathbb{Z}/l\mathbb{Z})^n$.

By algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n$ we mean a closed algebraic subvariety stable under the group operations. An irreducible algebraic subgroup is called a torus. Any algebraic subgroup is a finite disjoint union of translates of a torus by torsion points. Given an algebraic subgroup H we denote by H^0 its connected component containing $(1,\dots,1)$. For more on algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n$ see [11] or [27]

Let us recall that the essential minimum $\hat{\mu}^{\mathsf{ess}}(V)$ of a subvariety $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ is $\hat{\mu}^{\mathsf{ess}}(V) = \inf\{\theta > 0 \mid \overline{V(\theta)} = V\}$ where $V(\theta) = \{P \in V \mid \hat{h}(P) \le \theta\}$. Theorem 1.2 asserts that $\hat{\mu}^{\mathsf{ess}}(V) = 0$ if and only if V is a union of torsion varieties. The problem of finding sharp lower bounds for $\hat{\mu}^{\mathsf{ess}}(V)$ for a subvariety of $\mathbb{G}_{\mathrm{m}}^n$ which is not a union of torsion varieties is a generalization of Lehmer's problem ([21]). A lower bound for the essential minimum will depend on some geometric invariants of V, for instance its degree. Moreover, if we do not make any further geometric assumption on the variety, such a bound must also depend on its field of definition. Indeed, let H be a proper subtorus of $\mathbb{G}_{\mathrm{m}}^n$ and let $\boldsymbol{\alpha}_n$ be a sequence of non-torsion points whose height tends to zero (for instance, $\boldsymbol{\alpha}_n = (2^{1/n},\dots,2^{1/n})$). Then, the varieties $V_n = H\boldsymbol{\alpha}_n$ have fixed degree $\deg(H)$ and essential minimum $\hat{\mu}^{\mathsf{ess}}(V_n) \le \hat{h}(\boldsymbol{\alpha}_n) \to 0$. In spite of that, if we further assume that V is not a translate of a proper subtorus (even by a point of infinite order), then Bombieri and Zannier ([12]) proved that the essential minimum of V can be bounded from below only in terms of the degree of V.

Let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^n$. We define the obstruction index $\omega(V)$ of V as the minimum of $\deg(Z)$ where Z is a hypersurface containing V. As a trivial example, let α be an algebraic number of degree d and let $V \subset \mathbb{G}_{\mathrm{m}}$

be the set of its conjugates. Then $\omega(V) = d$. More generally, let V be an equidimensional subvariety of $\mathbb{G}_n^m$ of degree d. Then, by a result of Chardin (see [13])

$$\omega(V) \leq nd^{1/\mathrm{codim}(V)} \ . \tag{2.1}$$

It turns out that $\omega(V)$, and not the degree of V, is the right invariant to formulate the sharpest conjectures on $\hat{\mu}^{\mathsf{ess}}(V)$. Although, in order to get lower bounds for $\hat{\mu}^{\mathsf{ess}}(V)$ depending only on $\omega(V)$, we need some extra assumptions.

If we look for a lower bound of the essential minimum of a $\mathbb{Q}$-irreducible subvariety V (*arithmetic case*) depending only on $\omega(V)$, the assumption that V is not a union of torsion varieties is not sufficient. We need to assume moreover that V is *weak-transverse*, *i.e.* that V is not contained in a union of torsion varieties. To see that this further assumption is necessary, let us consider the family of subvarieties $V_l = \{(x, 1) \in \mathbb{G}_{\mathrm{m}}^2,\ x^l = 2\}$ ($l \in \mathbb{N}$) which are $\mathbb{Q}$-irreducibles, not torsion, and satisfy $\omega(V_l) = 1$ and $\hat{\mu}^{\mathsf{ess}}(V_l) = (\log 2)/l$.

Similarly, if we look for a lower bound of $\hat{\mu}^{\mathsf{ess}}(V)$ depending only on $\omega(V)$, where now V is an irreducible subvariety which may be not defined over $\mathbb{Q}$ (*geometric case*), the assumption that V is not a a translate of a subtorus is not sufficient. We need to assume that V is *transverse*, *i.e.* that V is not contained in a proper translate of a subtorus. As an example, consider a curve $\mathcal{C} \subseteq \mathbb{G}_{\mathrm{m}}^2$ which is not a translate of a subtorus. Let $\mathcal{C}' = \mathcal{C} \times \{1\} \subseteq \mathbb{G}_{\mathrm{m}}^3$ and choose, for $l \in \mathbb{N}$, an irreducible component V_l of $[l]^{-1}\mathcal{C}'$. Then $\hat{\mu}^{\mathsf{ess}}(V_l) \mapsto 0$, while $\omega(V_l) = 1$.

In [2] (arithmetic case) and [4] (geometric case) we made a conjectural lower bound for $\hat{\mu}^{\mathsf{ess}}(V)$ which will state now in a simplified version.

Conjecture 2.1. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a either a $\mathbb{Q}$-irreducible weak-transverse subvariety (arithmetic case) or an irreducible transverse subvariety (geometric case). Then there exists $c(n) \geq 1$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq (c(n)\omega(V))^{-1}.$$

This conjecture is proved "up to a remainder term" in both arithmetic case ([2] if $\dim V = 0$ and [3] if $\dim V > 0$) and geometric case ([4]). This is a simplified version of these results:

Theorem 2.2. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be as in Conjecture 2.1. Then for any $\varepsilon > 0$ there exists $c(n, \varepsilon) \geq 1$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq (c(n, \varepsilon)\omega(V))^{-1-\varepsilon}.$$

Remark that if $n = 1$ the arithmetic case of the conjecture reduces to Lehmer's conjecture, while the arithmetic case of the theorem is Dobrowolski's theorem (in a simplified version) [17]. Indeed the set of conjugates of an algebraic number α is weak-tranverse if and only if α is not a root of unity. For $n = 1$ the conjecture and the theorem are empty in the geometric case, since points are translate. For $n > 1$ the conjecture and the theorem are, in the arithmetic case, the natural generalization of Lehmer's conjecture and Dobrowolski's theorem. In this situation, theorem 2.2 sometimes produces lower bounds for the height of algebraic numbers which are even stronger than what is expected by Lehmer's conjecture. Let $\alpha_1, \ldots, \alpha_n$ algebraic numbers of height $\leq h$, lying in a number field of degree d. We remark that the 0-dimensional variety $V = \{\sigma(\boldsymbol{\alpha})\}_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} \subset \mathbb{G}_{\mathrm{m}}^n$ is weak-transverse if and only if $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent. Let us assume that this is the case. We have $\hat{\mu}^{\mathsf{ess}}(V) \leq h$ and, by (2.1),

$$\omega(V) \leq n d^{1/n}.$$

Thus, by the arithmetic part of theorem 2.2, for any $\varepsilon > 0$ we have

$$h \geq c(n, \varepsilon) d^{-1/n-\varepsilon}.$$

for some effective $c(n, \varepsilon) > 0$.

In [7] and [8] we gave a new and simpler proof of an improved a more explicit version of theorem 2.2. We will describe in details this new proof (in the geometric case) in the next sections. In the rest of this section we state some more conjectures and results concerning the localization of small points.

Let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^n$. As mentioned in the introduction, an equivalent version of theorem 1.2 says that the height on $V^*(\overline{\mathbb{Q}})$ is bounded from below by a positive quantity:

$$\mu^*(V) = \inf_{\boldsymbol{\alpha} \in V^*} \hat{h}(\boldsymbol{\alpha}) > 0.$$

Remark that obviously $\mu^*(V) \leq \hat{\mu}^{\mathsf{ess}}(V)$. Hence one could hope, in analogy to the conjectural lower (2.1), that, at least for a weak-transverse V defined over $\mathbb{Q}$,

$$\mu^*(V) \geq (c(n)\omega(V))^{-1}$$

for some constant $c(n) \geq 1$. This is false, as the following example shows. Let α_k be a sequence of algebraic numbers of minimal poynomial $f_k(t) \in \mathbb{Q}[t]$

whose height is positive and tends to zero as $k \to +\infty$ (thus $\deg(f_k) \to \infty$). Let us consider

$$V_k = \{(x_1, x_2, x_3) \in \mathbb{G}_{\mathrm{m}}^3 \mid f(x_1) = x_1^2 + x_1^3 - x_2 - x_3 = 0\}. \tag{2.2}$$

One checks (exercise 2.2) that V_k is weak-transverse, $\boldsymbol{\alpha}_k = (\alpha_k, \alpha_k^2, \alpha_k^3) \in V_k^*$, $h(\boldsymbol{\alpha}_k) \to 0$ and $\omega(V_k) \leq 3$.

Thus, we modify our guess as follow. Let $\delta(V)$ be the minimum integer δ such that V is the intersection of hypersurfaces $Z_1, \ldots, Z_r$ of degree $\leq \delta$. Assuming V defined over $\mathbb{Q}$, in [5] we conjecture a lower bound of the shape:

$$\mu^*(V) \geq (c(n)\delta(V))^{-1} ,$$

where as usual $c(n) \geq 1$. In the same article, we prove this conjecture up to a remainder term, as we state here in a simplified form:

Theorem 2.3. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a $\mathbb{Q}$-irreducible variety. Then for any $\varepsilon > 0$ there exists $c(n, \varepsilon) \geq 1$ such that*

$$\mu^*(V) \geq (c(n)\delta(V))^{-1-\varepsilon} .$$

We make a similar analysis for varieties not necessarily defined over $\mathbb{Q}$. Let V be a tranverse subvariety of $\mathbb{G}_{\mathrm{m}}^n$ and define, as in [12],

$$V^0 = V \setminus \bigcup_{B \subseteq V} B.$$

where the union is now on the set of translates B of tori of positive dimension. Again $V \setminus V^0$ is an open set (see [12] and [25]); Bombieri and Zannier prove that, outside a finite set, the height on V^0 is bounded from below by a positive quantity depending only on the ideal of definition of V and not on its field of definition. More precise result was obtained by Schmidt [25] and by David and Philippon (see [15])

Again we cannot expect bounds depending only on $\omega(V)$. In [6], the following is conjectured.

Conjecture 2.4. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety. There exists $c(n) \geq 1$ such that, for $\alpha \in V^0$ outside a finite subset of cardinality $\leq c(n)\delta(V)^n$, we have*

$$h(\boldsymbol{\alpha}) \geq (c(n)\delta(V))^{-1}.$$

Let $\varepsilon > 0$. Using a variant of the main result of [4] and an additional induction, in [6] theorem 1.5, we prove that, for all but finitely many $\boldsymbol{\alpha} \in V^0$,

$$\hat{h}(\boldsymbol{\alpha}) \geq (c(n,\varepsilon)\delta(V))^{-1-\varepsilon},$$

where $c(n,\varepsilon) \geq 1$. We remark that the proof of [6] can not produce a bound of the conjectured shape for the cardinality of the set of points of pathologically small height. This last gap was filled in [8] where we prove the above conjecture up to a remainder term, as we state now in a simplified version.

Theorem 2.5. *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety. Then for any $\varepsilon > 0$ there exists $c(n,\varepsilon) \geq 1$ such that, for $\alpha \in V^0$ outside a finite subset of cardinality $\leq (c(n,\varepsilon)\delta(V))^{n+\varepsilon}$, we have*

$$h(\boldsymbol{\alpha}) \geq (c(n,\varepsilon)\delta(V))^{-1-\varepsilon}.$$

Exercice 2.1. *Prove properties i)-iv) of Weil's height.*

Exercice 2.2. *Let V_k as in (2.2). Show that V_k is weak-transverse, $\boldsymbol{\alpha}_k = (\alpha_k, \alpha_k^2, \alpha_k^3) \in V_k^*$, $h(\boldsymbol{\alpha}_k) \to 0$ and $\omega(V_k) \leq 3$.*

Exercice 2.3. *Let B be a translate of a subtorus and let $\varepsilon \geq 0$. Prove that $B(\varepsilon)$ is either empty or dense in B.*

Exercice 2.4. *Let $V \subsetneq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety and let $\varepsilon > 0$. Assume that $V(\varepsilon)$ is contained in a finite union of translates of subtori contained in V. Prove that $\overline{V(\varepsilon)}$ is the union of some of these translates.*

Exercice 2.5. *Let V be a $\mathbb{Q}$-irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Prove that there exists a hypersurface Z* **defined over** $\mathbb{Q}$*, containing V and of degree $\omega(V)$.*

3 Overview of the methods

The original proofs of the lower bound for the essential minimum in the arithmetic and geometric case require several technical tools. By contradiction, we assume in both proofs that the essential minimum is sufficiently small. We then start following the usual steps of a transcendence proof:

interpolation (construction of an auxiliary function), extrapolation and zero estimates. Concerning the zero lemma, in both cases these proofs become very technical.

Let us consider first the arithmetic case. In diophantine analysis a classical zero lemma (as the main result of [22]) is normally enough to conclude the proof. On the contrary, in [2] we need a more complicated zero lemma. As a consequence, this force to extrapolate over different set of primes. In Dobrowolski's proof ([17]) one construct, using Siegel's Lemma, an auxiliary function F which vanishes on α. Then we extrapolate by proving that F must also vanish on α^p at least for small primes p. In the proof of the arithmetic case of theorem 2.2 we construct an auxiliary function vanishing on V and then we extrapolate by proving that F must also vanish on $[l]V$ for any product $l = p_1 \cdots p_k$ of small primes ($k = \operatorname{codim}(V)$). The zero lemma we alluded before shows that for some l the obstruction index ω' of $\{\sigma(\boldsymbol{\alpha}^l)\}_{\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ is pathologically small than the obstruction index ω of $\{\sigma(\boldsymbol{\alpha})\}_{\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$. Unfortunately, it seems hard to find a lower bound for ω' in terms of ω. Thus, we cannot conclude easily the proof. To avoid this problem, we start again the whole construction replacing V with $[l]V$. To ensure that the process end at some moment, we need a cumbersome induction (*descent step*).

The situation is quite similar in the original proof of the lower bound for the essential minimum in the geometric case. In this proof we construct again an auxiliary function vanishing on V and then we extrapolate by proving that F must also vanish on $\ker[l]V$ for l as before. We need again a variant of a zero lemma which use the fact that our set of translation (the union of $\ker[l]$) is actually an union of big subgroups. Using this new zero lemma we succeed to show that for some l the obstruction index $\omega([l]V)$ is pathologically small than $\omega(V)$. As in the arithmetic situation, we conclude the proof with a descent step.

In [7] we simplify the proof of the geometric result, giving a totally explicit statement:

Theorem 3.1. *Let $V \subset \mathbb{G}_{\mathrm{m}}^n$ be an irreducible transverse variety of codimension k. Then*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \omega(V)^{-1}\big(200n^5\log(n^2\omega(V))\big)^{-nk^2}.$$

The proof consists of several steps. First we code the diophantine analysis in an inequality involving some parameters, the essential minimum of a subvariety of $\mathbb{G}_{\mathrm{m}}^n$ and two Hilbert's functions. The key idea to decode the

diophantine information is to use sharp estimates for the Hilbert function. The upper bound is a variant of the main result of [13]. It is proved in [4], lemma 2.5. The lower bound is a deep result of Chardin and Philippon [14], corollary 3. Using these tools, in[1] [7] we prove:

Theorem 3.2. *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$ of codimension k which is not a translate of a torus. Let*

$$\theta_0 = \delta_0(V)\big(27n^2 \log(n^2\delta_0(V))\big)^{kn}.$$

Then $V(\theta_0^{-1})$ is contained in a hypersurface Z of degree at most θ_0 which does not contain V. In particular, $\hat{\mu}^{\mathsf{ess}}(V) \geq \theta_0^{-1}$.

In this theorem $\delta_0(V)$ is the minimal degree δ_0 such that V is an irreducibile component of an intersection of hypersurfaces of degree $\leq \delta_0$. We recall that $\delta(V)$ has been defined as the minimal degree δ such that V is, as a set, intersection of hypersurfaces of degree $\leq \delta$. If V is irreducible, then

$$\omega(V) \leq \delta_0(V) \leq \delta(V) \leq \deg(V) \leq \delta_0(V)^{\mathrm{codim}(V)}. \tag{3.1}$$

The first three inequalities are immediate. The last one follows from [22] proposition 3.3 with $p = 1$, $N_1 = n$ and $D_1 = \delta_0(V)$.

A priori, it is difficult to compare theorem 3.2 with theorem 3.1. On the one hand, in theorem 3.2 we do not assume that V is transverse, but only that V is not a translate of a torus. On the other hand, the bound in theorem 3.2 depends on $\delta_0(V)$ which could potentially be equal to the degree of V, while

$$\omega(V) \leq n \deg(V)^{1/\mathrm{codim}(V)}.$$

Nevertheless, a new reduction process (originated by an idea of Viada) applied to *each* variety involved, allows us to deduce theorem 3.1 from theorem 3.2.

In the next four sections, which are largely inspired by [7], we shall describe in detail the main steps of this method. In section 4 we code the diophantine information. In section 5 we prove theorem 3.2. In section 6 we deduce theorem 3.1. In the last section we come back to the localization of small points, proving a general result (theorem 7.3) which implies theorem 2.5.

[1] A preliminary version of this theorem already appears [1].

4 Diophantine analysis: encoding the information

The original proof of the lower bound for the essential minimum in the geometric case relies on the fact that V is p-adically close to $\boldsymbol{\zeta}V$ for all p-torsion points $\boldsymbol{\zeta}$ and for all *small* primes p. This is a consequence of the following simple remark. For any p-root of unity ζ and for any place $v \mid p$ we have $|1-\zeta|_v \leq p^{-1/(p-1)}$. But also *all* the translates of V by p-torsion points are p-adically close to each other. This gives a first simplification: we replace the vanishing principle used in [4] by a *symmetric* vanishing principle. For technical reasons, it is more convenient to use an interpolation determinant than an auxiliary function. In this new diophantine approach, the Hilbert function appears in a natural way. Let us recall some basic facts on it.

Let $I \subset \overline{\mathbb{Q}}[\mathbf{x}]$ be a homogeneous reduced ideal. For $\nu \in \mathbb{N}$ we denote by $H(I;\nu)$ the Hilbert function $\dim[\overline{\mathbb{Q}}[\mathbf{x}]/I]_\nu$. Let T be a positive integer and $I \subset \overline{\mathbb{Q}}[\mathbf{x}]$ be a homogeneous reduced ideal. We denote by $I^{(T)}$ the T-symbolic power of I, *i.e.* the ideal of polynomials vanishing on the variety defined by I with multiplicity at least T. Let V be a variety of $\mathbb{G}_{\mathrm{m}}^n$, defined in $\mathbb{P}_n$ by a reduced ideal I. By abuse of notations, we set $H(V;\nu) = H(I;\nu)$ and $H(V,T;\nu) = H(I^{(T)};\nu)$.

We can now proceed on the first part of the proof: we *encode* the diophantine information in a general lower bound involving some parameters and two related Hilbert's function.

Theorem 4.1. *Let ν, T be positive integers and let p be a prime number. Let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Then*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \left(1 - \frac{H(V,T;\nu)}{H(\ker[p]V;\nu)}\right)\frac{T\log p}{p\nu} - \frac{n}{2\nu}\log(\nu+1).$$

Proof. Choose any real ε such that $\varepsilon > \hat{\mu}^{\mathsf{ess}}(V)$. For simplicity we define $S = V(\varepsilon)$. Then S is Zariski dense in V. We consider the (potentially infinite) matrix $(\boldsymbol{\beta}^{\boldsymbol{\lambda}})$ where the lines are indexed by[2] $\boldsymbol{\beta} \in \ker[p]S$ and the columns by the vectors $\boldsymbol{\lambda} \in \mathbb{N}^{n+1}$ with $|\lambda| = \lambda_0 + \cdots + \lambda_n = \nu$. Since S is Zariski dense in V, this matrix has rank $L = H(\ker[p]V;\nu)$. We select $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_L \in \ker[p]S$ and $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L$ with $|\boldsymbol{\lambda}_j| = \nu$ such that

$$\det(\boldsymbol{\beta}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\ldots,L} \neq 0\ .$$

[2] we identify $\boldsymbol{\beta} \in \mathbb{G}_{\mathrm{m}}^n$ with $(1,\boldsymbol{\beta}) \in \mathbb{P}_n$

Thus the multi-homogeneous polynomial

$$F(\mathbf{x}_1,\dots,\mathbf{x}_L)=\det(\mathbf{x}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\dots,L}\in\mathbb{Z}[\mathbf{x}_1,\dots,\mathbf{x}_L]$$

does not vanish at $(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_L)$. Choose $\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L\in S$ such that $\boldsymbol{\beta}_j\in\ker[p]\boldsymbol{\alpha}_j$. We have

Lemma. *The polynomial F vanishes on $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L)$ with multiplicity at least $T_0=\big(L-H(V,T;\nu)\big)T$.*

Proof. Let $W=\{\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L\}$. Obviously we may assume $H(V,T;\nu)<L$. Let $L_0=L-H(W,T;\nu)$ and remark that $L_0\geq L-H(V,T;\nu)>0$. Let $E_1,E_2\subseteq\overline{\mathbb{Q}}[x_0,\dots,x_n]_\nu$ be respectively the vector space generated by $\mathbf{x}^{\boldsymbol{\lambda}_1},\dots,\mathbf{x}^{\boldsymbol{\lambda}_L}$ and the vector space of homogeneous polynomials of degree ν vanishing on W with multiplicity at least T. Since $F\neq 0$, we have $\boldsymbol{\lambda}_i\neq\boldsymbol{\lambda}_j$ for $i\neq j$; thus $\dim(E_1)=L$. By definition of E_2 we have $\dim(E_2)=\binom{n+\nu}{n}-H(W,T;\nu)$. Then

$$\begin{aligned}\dim(E_1\cap E_2)&=\dim(E_1)+\dim(E_2)-\dim(E_1+E_2)\\&=L+\binom{n+\nu}{n}-H(W,T;\nu)-\dim(E_1+E_2)\geq L_0\ .\end{aligned}$$

Thus, there exist L_0 linearly independent polynomials

$$G_1=\sum_{j=1}^{L}g_{1j}\mathbf{x}^{\boldsymbol{\lambda}_j},\dots,G_{L_0}=\sum_{j=1}^{L}g_{L_0j}\mathbf{x}^{\boldsymbol{\lambda}_j}$$

vanishing on W with multiplicity $\geq T$. Without loss of generality we can assume

$$\det(g_{k,j})_{\substack{1\leq k\leq L_0\\ L-L_0<j\leq L}}\neq 0\ .$$

Thus we may replace the last L_0 columns of the matrix $(\mathbf{x}_i^{\boldsymbol{\lambda}_j})$ by

$${}^{\tau}\big(G_k(\mathbf{x}_1),\dots,G_k(\mathbf{x}_L)\big)\ ,\qquad k=1,\dots,L_0$$

changing its determinant only by a nonzero constant. Since the polynomials G_k vanish on W with multiplicity $\geq T$, developing the determinant with respect to the new last L_0 columns we see that it vanishes on $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L)$ with multiplicity $\geq L_0T\geq T_0$ as required.

$\square$

Let v be a place dividing p. Using the inequality $|1-\zeta_p|_v \le p^{-1/(p-1)}$ we get

$$|\boldsymbol{\alpha}_{j,k}-\boldsymbol{\beta}_{j,k}|_v \le p^{-1/(p-1)}|\boldsymbol{\alpha}_{j,k}|_v$$

for $j=1,\dots,L$ and $k=1,\dots,n$. We perform the Taylor expansion of F around $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L)$. Since F vanishes on $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_L)$ with multiplicity at least T_0 we obtain

$$|F(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_L)|_v \le p^{-T_0/(p-1)}\prod_{j=1}^{L}|\boldsymbol{\alpha}_j|_v^{\nu}\ .$$

where $|\boldsymbol{\alpha}_k|_v=\max\{1,|\alpha_{j,1}|_v,\dots,|\alpha_{j,n}|_v\}$.

By the ultrametric inequality for $v\nmid\infty$ and by the Hadamard inequality for $v\mid\infty$ we obtain that, for an arbitrary place v,

$$|F(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_L)|_v \le \begin{cases}\prod_{j=1}^{L}|\boldsymbol{\beta}_j|_v^{\nu}, & \text{if } v\nmid\infty\\ L^{L/2}\prod_{j=1}^{L}|\boldsymbol{\beta}_j|_v^{\nu}, & \text{if } v\mid\infty\ .\end{cases}$$

Since $\boldsymbol{\alpha}_k$ is a translate of $\boldsymbol{\beta}_k$ by a torsion point, $|\boldsymbol{\beta}_k|_v=|\boldsymbol{\alpha}_k|_v$. We apply the product formula to $F(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_L)\neq 0$. Let K be a sufficiently large number field.

$$\begin{aligned}0 &\le \sum_{v\in\mathcal{M}_K}\frac{[K_v:\mathbb{Q}_v]}{[K:\mathbb{Q}]}\log|F(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_L)|_v\\ &\le -\frac{T_0\log p}{p-1}+\frac{L}{2}\log L+\nu\sum_{j=1}^{L}h(\boldsymbol{\alpha}_j)\le -\frac{T_0\log p}{p}+\frac{L}{2}\log L+\nu L\varepsilon\ .\end{aligned}$$

Moreover $L\le(\nu+1)^n$. Thus

$$\varepsilon\ge\frac{T_0\log p}{Lp\nu}-\frac{n}{2\nu}\log(\nu+1)\ .$$

Taking the limit for ε which tends to $\hat{\mu}^{\mathsf{ess}}(V)$ we obtain the wished bound.

□

5 Proof of theorem 3.2: decoding the information

A technical part of the proof of theorem 3.2 is devoted to the computation of the constant involved. In order to simplify things and to improve the comprehension, we restrict ourself to sketch a proof of a non explicit version of theorem 3.2.

Theorem 5.1. *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$ of codimension k which is not a translate of a torus. Then, there exists $\theta_0 > 0$ with*

$$\theta_0 \leq c(n)\delta_0(V)\big(\log(n^2\delta_0(V))\big)^{kn}.$$

for some $c(n) \geq 1$ and such that $V(\theta_0)$ is contained in a hypersurface Z of degree at most θ_0 which does not contain V (in particular, this implies $\hat{\mu}^{\mathsf{ess}}(V) \geq \theta_0^{-1}$).

Proof. Let $d = n - k = \dim(V)$ and $\delta_0 = \delta_0(V)$. From now on, we use in a non standard way the symbols $\approx$, $\ll$ and $\gg$. We write $A \approx B$ if and only if $c_1 B < A < c_2 B$ with c_1, $c_2 > 0$. The constants c_1,c_2 are eventually assumed to be sufficiently large (or small) in such a way that the forthcoming assumptions are verified. Similarly, $A \ll B$ (or $B \gg A$) if and only if $A \leq cB$ where $c > 0$ has the same meaning as before.

Let $\theta_0 > 0$ which will be fixed later. Let W be the Zariski closure of the set $V(\theta_0^{-1})$ and let $W' = \ker[p]W$. Then, $\hat{\mu}^{\mathsf{ess}}(W) \leq \theta_0^{-1}$. By theorem 4.1 (with V replaced by W and ν replaced by θ_0),

$$\theta_0^{-1} \geq \hat{\mu}^{\mathsf{ess}}(W) \geq \left(1 - \frac{H(W,T;\theta_0)}{H(W';\theta_0)}\right)\frac{T\log p}{p\theta_0} - \frac{n}{2\theta_0}\log(\theta_0+1)$$

As $W \subseteq V$ we have $H(W,T;\theta_0) \leq H(V,T;\theta_0)$. We want to show that $H(W';\theta_0) < H(V';\theta_0)$. Assume by contradiction that $H(W';\theta_0) \geq H(V';\theta_0)$. Thus

$$\theta_0^{-1} \geq \hat{\mu}^{\mathsf{ess}}(W) \geq \left(1 - \frac{H(V,T;\theta_0)}{H(V';\theta_0)}\right)\frac{T\log p}{p\theta_0} - \frac{n}{2\theta_0}\log(\theta_0+1) \qquad (5.1)$$

We recall that *for large* ν the Hilbert function $H(V;\nu)$ is actually a polynomial of degree $\dim(V)$ and leading coefficient $\deg(V)/\dim(V)!$. To go on with the proof, we need a lower bound for $H(V';\theta_0)$ and un upper bound for $H(V,T;\theta_0)$. Fortunately, both are at our disposal. Let

$$m = k\big(\delta_0(V') - 1\big)\ .$$

By a (deep) result of M. Chardin and Philippon ([14], corollary 3) we have, if $\theta_0 > m$,

$$H(V';\theta_0) \geq \binom{\theta_0 + d - m}{d} \deg(V'). \tag{5.2}$$

On the other hand, from a result of M. Chardin [13] (see lemma 2.5 of [4] for details), for any $\theta_0 > 0$,

$$H(V,T;\theta_0) \leq \binom{T-1+k}{k}\binom{\theta_0+d}{d} \deg(V). \tag{5.3}$$

To exploit (5.2) we still need to compare $\delta_0(V')$ with $\delta_0(V)$ and $\deg(V')$ with $\deg(V)$. These is the object of the following technical lemma whose proof is omitted (see [7], lemma 3.8):

Lemma. *Let t be the number of irreducible components of V'. Then*

$$\deg(V') = t\deg(V) \quad \textit{and} \quad \delta_0(V') \leq t\delta_0(V).$$

Now we need some bounds for t. We recall the definition of stabilizer:

$$\mathrm{Stab}(V) = \{\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n \mid \boldsymbol{\alpha}V = V\} = \bigcap_{\mathbf{x}\in V} \mathbf{x}^{-1}V\ .$$

Thus $\mathrm{Stab}(V)$ is an algebraic subgroup of dimension $\leq \dim(V)$. We remark that we have equality of dimensions if and only if V is a translate of a torus. Since $\mathrm{Ker}([p])V = \cup_{\boldsymbol{\zeta}\in\mathrm{Ker}[p]}\boldsymbol{\zeta}V$ we have $t = p^n|\mathrm{Ker}([p]) \cap \mathrm{Stab}(V)|^{-1}$. Remark that

$$|\mathrm{Ker}([p]) \cap \mathrm{Stab}(V)| = p^{\dim \mathrm{Stab}(V)}|\mathrm{Ker}([p]) \cap (\mathrm{Stab}(V)/\mathrm{Stab}(V)^0)|.$$

Thus, if $p \nmid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$, then $t = p^{\mathrm{codim}(\mathrm{Stab}\, V)}$. By [18], lemma 8 and by (3.1) we have

$$\deg(\mathrm{Stab}(V)) \leq \deg(V)^{\dim(V)+1} \leq \delta_0^{nk}$$

which implies

$$\log[\mathrm{Stab}(V) : \mathrm{Stab}(V)^0] \leq nk\log(\delta_0). \tag{5.4}$$

Let

$$N \approx (\log(n^2\delta_0))^k.$$

If for any prime p with $N/2 \leq p \leq N$ we have $p \mid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$ then

$$\log[\mathrm{Stab}(V) : \mathrm{Stab}(V)^0] \geq \sum_{N/2\leq p\leq N} \log p \gg N. \tag{5.5}$$

Equations (5.4) and (5.5) are not consistent. We conclude that there exists a prime $p \nmid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$ satisfying $N/2 \le p \le N$. We choose a such prime p. Thus

$$t = p^{\mathrm{codim}(\mathrm{Stab}\, V)} \ .$$

Since V is not a translate of a torus, $k+1 \le \mathrm{codim}(\mathrm{Stab}\, V) \le n$. By the lemma above,

$$\deg(V') = t \deg(V) \ge p^{k+1} \deg(V) \quad \text{and} \quad \delta_0(V') \le t\delta_0(V) \le p^n \delta_0. \tag{5.6}$$

The upper bound for $\delta_0(V')$ in (5.6) gives

$$m \le k p^n \delta_0 \ .$$

Choose

$$\theta_0 = md + m \quad \text{and} \quad T \approx p^{1+1/k}.$$

Thus $\theta_0 > m$ as required for (5.2) and

$$\theta_0 \ll \delta_0 \log(n^2 \delta_0)^{kn} \tag{5.7}$$

as required in the statement of theorem 5.1. By (5.2) and (5.3) we get

$$\frac{H(V,T;\theta_0)}{H(V';\theta_0)} \le \frac{\binom{T-1+k}{k}\binom{\theta_0+d}{d}\deg(V)}{\binom{\theta_0+d-m}{d}\deg(V')}.$$

By the lower bound for $\deg(V')$ given in (5.6) and by the choices of θ_0 and T,

$$\frac{\binom{T-1+k}{k}\binom{\theta_0+d}{d}\deg(V)}{\binom{\theta_0+d-m}{d}\deg(V')} \le \frac{\binom{T-1+k}{k}\binom{\theta_0+d}{d}}{\binom{\theta_0+d-m}{d}p^{k+1}} \ll \frac{T^k}{p^{k+1}}\left(1+\frac{m}{\theta_0-m}\right)^d < \frac{1}{2},$$

say. Inserting these inequalities in (5.1) we finally obtain

$$\begin{aligned} \theta_0^{-1} &\ge \left(1 - \frac{H(V,T;\theta_0)}{H(V';\theta_0)}\right)\frac{T\log p}{p\theta_0} - \frac{n}{2\theta_0}\log(\theta_0+1) \\ &\ge \left(\frac{T\log p}{2p} - \frac{n}{2}\log(\theta_0+1)\right)\theta_0^{-1}. \end{aligned} \tag{5.8}$$

We have

$$\frac{T\log p}{2p} \gg p^{1/k}\log p \gg \log(n^2\delta_0)$$

and, by (5.7),

$$\log(\theta_0+1) \ll \log(n^2\delta_0).$$

Thus, choosing in an appropriate way the implicit constants in the parameters,

$$\frac{T\log p}{2p} - \frac{n}{2}\log(\theta_0+1) > 1$$

and (5.8) is inconsistent. This contradiction shows that we cannot have $H(W';\theta_0) \geq H(V';\theta_0)$. Thus $H(W';\theta_0) < H(V';\theta_0)$. Since $W' \subseteq V'$ we infer that there exists a homogeneous polynomial F of degree $\leq \theta_0$ vanishing on W' (and thus on $V(\theta_0^{-1})$) but not on V'. Replacing $F(\mathbf{x})$ by $F(\boldsymbol{\zeta}\mathbf{x})$ for a suitable $\boldsymbol{\zeta} \in \ker[p]$, we can assume $F \neq 0$ on V (recall that W' is invariant by translation by p torsion points).

□

6 Proof of theorem 3.1: from $\delta_0(V)$ to $\omega(V)$.

In this section we deduce theorem 3.1 from the apparently weaker theorem 3.2. We let

$$\omega = \omega(V) \quad \text{and} \quad \theta = \omega\big(200n^5\log(n^2\omega)\big)^{nk^2}.$$

We assume by contradiction that $V(\theta^{-1})$ is Zariski dense in V. For $r \in \{1,\dots,k\}$, let

$$D_r = \omega\big(200n^5\log(n^2\omega)\big)^{rkn}$$

and remark that $D_r \leq \theta$. We construct by induction a chain of varieties

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_r \supseteq \cdots \supseteq X_k$$

satisfying for $r = 0,\dots,k$ the following

Claim$_r$

i) $V \subseteq X_r$;

ii) each irreducible component of X_r containing V has codimension $\geq r+1$;

iii) $\delta(X_r) \leq D_r$.

Theorem 3.1 is proved if we show that $\mathbf{Claim}_k$ holds. Indeed, by $\mathbf{Claim}_k$ i) there exists an irreducible component W of X_r which contains V. By ii) $k = \mathrm{codim}(V) \geq \mathrm{codim} W \geq k+1$. This gives a contradiction.

We now start our inductive construction.

• For $r = 0$, we choose for X_0 a geometrically irreducible hypersurface containing V of minimal degree ω. As X_0 is a hypersurface, $\delta_0(X_0) = \delta(X_0) = \deg X_0 = \omega$. Assertions i), ii) and iii) clearly hold.

• We assume that $\mathbf{Claim}_r$ holds for some $r \in \{0, \ldots, k-1\}$ and we prove that $\mathbf{Claim}_{r+1}$ still holds. Let X_r be a Zariski closed set satisfying conditions i), ii) and iii) of $\mathbf{Claim}_r$. Since $V \subseteq X_r$ there exists at least one irreducible component of X_r which contains V. Let $W_1, \ldots, W_s$ be the irreducible components of X_r which contain V and let $W_{s+1}, \ldots, W_t$ those which do not contain V.

Let $j \in \{1, \ldots, s\}$. Since $\delta(X_r) \leq D_r$, the variety W_j is an irreducible component of an intersection of hypersurfaces of degree $\leq D_r$. Thus $\delta_0(W_j) \leq D_r$. Moreover V is transverse and $V \subseteq W_j$. Thus W_j is not a translate of a subtorus. Let

$$\theta_0 = D_r \Big(27 n^2 \log \big(n^2 D_r \big) \Big)^{kn}.$$

By theorem 3.2 the set $W_j(\theta_0^{-1})$ is contained in a hypersurface Z_j which does not contain W_j and such that $\deg Z_j \leq \theta_0$. Furthermore

$$\theta_0 \ll \delta \big(\log(n^2 \delta) \big)^{r k_0 n + k_0 n} \ll D_{r+1}$$

and a more precise computation shows that $\theta_0 \leq D_{r+1}$. Since $V \subseteq W_j$ and $\theta_0 \leq D_{r+1} \leq \theta$, we have $V(\theta^{-1}) \subseteq W_j(\theta_0^{-1}) \subseteq Z_j$. By assumption $V(\theta^{-1})$ is Zariski dense in V. Thus, $V \subseteq Z_j$ for $j = 1, \ldots, s$.

Let

$$X_{r+1} = X_r \cap Z_1 \cap \cdots \cap Z_s.$$

We show that X_{r+1} satisfy $\mathbf{Claim}_{r+1}$. Since $V \subseteq X_r$ and $V \subseteq Z_j$ for $j = 1, \ldots, s$ we have $V \subseteq X_{r+1}$. This shows assertion i). Since $\deg Z_j \leq \theta_0 \leq D_{r+1}$ for $j = 1, \ldots, s$, we have $\delta(X_{r+1}) \leq \max\{\delta(X_r), D_{r+1}\} \leq \max\{D_r, D_{r+1}\} = D_{r+1}$, as required by iii). Let us show now assertion ii). We decompose X_{r+1} as

$$X_{r+1} = W_1' \cup \cdots \cup W_s' \cup W_{s+1}' \cup \cdots \cup W_t',$$

where $W_j' = W_j \cap Z_1 \cap \cdots \cap Z_s$. For $j = s+1, \ldots, t$ we have $V \not\subseteq W_j$ and thus the variety V is not contained in any irreducible component of W_j'. For

$j \in \{1, \ldots, s\}$ we have $W_j \not\subseteq Z_j$. Moreover, by $\mathbf{Claim}_r$ ii), $\operatorname{codim}(W_j) \geq r$. Thus every irreducible component of W_j' has codimension $> \operatorname{codim}(W_j)+1 \geq r+1$ as required.

7 Small points

Without any additional effort, the method of the proof of theorem 3.1 can be easily modified to get the following result (see [7], theorem 2.2):

Theorem 7.1. *Let $V_0 \subseteq V_1$ be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ of codimensions k_0 and k_1 respectively. Assume that V_0 is irreducible. Let*

$$\theta = \delta(V_1)\big(200n^5 \log(n^2\delta(V_1))\big)^{(k_0-k_1+1)k_0 n}.$$

Then,

- *either there exists a translate B of a torus such that $V_0 \subseteq B \subseteq V_1$ and $\delta_0(B) \leq \theta$,*
- *or there exists a hypersurface Z of degree at most θ such that $V_0 \not\subseteq Z$ and $V_0(\theta^{-1}) \subseteq Z$.*

Note that theorem 3.1 becomes a corollary of this theorem (choose $V_0 = V$ and V_1 an irreducible hypersurface of degree $\omega(V)$ containing V). Moreover, theorem 3.1 immediately implies an improved and explicit version of theorem 1.5 of [6]. Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be an irreducible variety of codimension k. Define

$$\theta = \delta(V)\big(200n^5 \log(n^2\delta(V))\big)^{k^2 n}.$$

Let V_0 be one of the finitely many irreducible components of

$$W = \overline{V(\theta^{-1})}.$$

Then $\overline{V_0(\theta^{-1})} = V_0$. Apply theorem 7.1 to the irreducibile component V_0 and to $V_1 = V$. Then, V_0 is contained in a translate B of a torus such that $B \subseteq V$ and $\delta_0(B) \leq \theta$. Varing V_0 over all irreducibile components of W, we conclude that $W \subseteq \cup B_j$ where $B_j \subseteq V$ are translates of tori with $\delta_0(B_j) \leq \theta$. In particular, for all but finitely many $\boldsymbol{\alpha} \in V^0$, we have $\hat{h}(\boldsymbol{\alpha}) \geq \theta^{-1}$.

Theorem 7.1 has the following arithmetic counterpart ([8], theorem 1.3), which implies the lower bounds for $\hat{\mu}^{\mathsf{ess}}(V)$ in the arithmetic case and for $\mu^*(V)$ stated respectively in theorem 2.2 and theorem 2.3 (see exercise 7.5 and exercise 7.6).

Theorem 7.2. *Let $V_0 \subseteq V_1$ be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$, defined over $\mathbb{Q}$, of codimensions k_0 and k_1 respectively. Assume that V_0 is $\mathbb{Q}$-irreducible. Let*

$$\theta = \delta(V_1)\big(935n^5 \log(n^2\delta(V_1))\big)^{(k_0-k_1+1)(k_0+1)(n+1)} .$$

Then,

- *either there exists a $\mathbb{Q}$-irreducible B union of torsion varieties such that $V_0 \subseteq B \subseteq V_1$ and $\delta_0(B) \leq \theta$,*
- *or there exists a hypersurface Z defined over $\mathbb{Q}$ of degree at most θ such that $V_0 \nsubseteq Z$ and $V_0(\theta^{-1}) \subseteq Z$.*

In the geometric case a further induction (suggested by Viada) leads us to the following result ([7] theorem 1.2).

Theorem 7.3. *Let $V \subsetneq \mathbb{G}_{\mathrm{m}}^n$ be a variety of codimension k. We decompose V as a (reduced) union $X_k \cup \cdots \cup X_n$, where X_j is an equidimensional variety of codimension[3] j. We define*

$$\theta = \delta(V)\big(200n^5 \log(n^2\delta(V))\big)^{(n-k)n(n-1)} .$$

Then,

$$\overline{V(\theta^{-1})} = G_k \cup \cdots \cup G_n$$

where G_j is either the empty set or a finite union of translates $B_{j,i}$ of subtori of codimension j such that $\delta_0(B_{j,i}) \leq \theta$. Moreover, for $r = k, \ldots, n$,

$$\sum_{i=k}^{r} \theta^{r-i} \deg G_i \leq \sum_{i=k}^{r} \theta^{r-i} \deg X_i \leq \theta^r . \tag{7.1}$$

Remark that this theorem immediately implies (up to a log factor) conjecture 2.4.

We now describe the inductive process which allow to deduce theorem 7.3 from theorem 7.1. We construct:

• $G_1, \ldots, G_{n-1}$ such that each G_j is a finite (possibly empty) union of translates $B_{j,i}$ of subtori of codimension j and such that $\delta_0(B_{j,i}) \leq \theta$;

[3] We allow the empty set as an equidimensional variety of arbitrary codimension with no irreducibile components and degree zero.

• equidimensional subvarieties $X'_k, \dots X'_n$ of codimension $k, \dots, n$

such that the following claim holds for $r \in \{k, \dots, n\}$

Claim$_r$

i) if $r > k$ then G_{r-1} is a union of irreducible components of X'_{r-1};

ii) $V(\theta^{-1}) \subseteq G_k \cup \dots \cup G_{r-1} \cup X'_r \cup X_{r+1} \cup \dots \cup X_n$;

iii) $\sum_{i=k}^{r-1} \theta^{r-i} \deg G_i + \deg X'_r \leq \sum_{i=k}^{r} \theta^{r-i} \deg X_i$.

Theorem 7.3 follows from our claim setting $G_n := X'_n$ (which is again a union of translates of subsubtori since it is a zero dimensional variety). Indeed, by assertion ii) of **Claim**$_n$, $V(\theta^{-1}) \subseteq G_k \cup \dots \cup G_n$. By exercise 2.4 we may assume

$$\overline{V(\theta^{-1})} = G_k \cup \dots \cup G_n \ .$$

We still have to prove (7.1) for $r = k, \dots, n$. Let $r \in \{k, \dots, n\}$. If $r < n$ we have $\deg(G_r) \leq \deg(X'_r)$ by **Claim**$_{r+1}$ i). This is still true if $r = n$ by the choice of G_n. Thus assertion iii) of **Claim**$_r$ implies the first inequality of (7.1). Corollary 5 of [23] (with $m = n$ and $S = \mathbb{P}^n$) shows that for $\theta \geq \delta(V)$ we have

$$\sum_{i=k}^{r} \theta^{r-i} \deg X_i \leq \theta^r \ ,$$

which gives the second inequality of (7.1).

It remains to construct by induction the varieties G_j and X'_j and to prove our claim.

We choose $X'_k = X_k$. **Claim**$_r$ is obviously satisfied. Let now $r \in \{k, \dots, n-1\}$ and suppose to have already constructed $G_k, \dots, G_{r-1}, X'_r$ which satisfy **Claim**$_r$. We want to construct G_r and X'_{r+1} in such a way that **Claim**$_{r+1}$ is satisfied. We first remark that we may assume:

a) No irreducible component of X'_r is imbedded in $G_k \cup \dots \cup G_{r-1}$;

b) Every irreducible component of X'_r meets $V(\theta^{-1})$.

This is clear because we can discard the irreducible components of X'_r not satisfying a) or b) without changing ii) and iii) of **Claim**$_r$. We then decompose X'_r as

$$X'_r = G_r \cup W_1 \cup \dots \cup W_s \tag{7.2}$$

where G_r is the union of the irreducible components $B_{r,i}$ of X'_r which are translates of subtori and such that $\delta_0(B_{r,i}) \leq \theta$ (possibly $G_r = \emptyset$) and where $W_1, \dots, W_s$ are the other irreducible components of X'_r.

Using conditions a), b) and exercise 2.4, it is easy to see (exercise 7.8) that

Remark 7.4. *Let B be a translate of a subtorus such that $W_i \subseteq B \subseteq V$ for some $i \in \{1, \dots, s\}$. Then $\delta_0(B) > \theta$.*

Let $i \in \{1, \dots, s\}$. We apply theorem 7.1 to the varieties $V_0 = W_i$ and $V_1 = V$. We have $k_0 = r \leq n-1$ and $k_1 = k$. The first conclusion of that theorem cannot occur, because of the previous remark. Thus, the second conclusion must hold. Namely, there exists a hypersurface Z_i of degree $\leq \theta$ such that $W_i \not\subseteq Z_i$ and $W_i(\theta^{-1}) \subseteq Z_i$. By Krull's Hauptsatz, $W_i \cap Z_i$ is either the empty set or it is an equidimensional variety of codimension $r+1$. We define

$$X'_{r+1} = X_{r+1} \cup \bigcup_{i=1}^{s} (W_i \cap Z_i) \ .$$

By construction,

$$V(\theta^{-1}) \subseteq G_k \cup \dots \cup G_r \cup X'_{r+1} \cup X_{r+2} \cup \dots \cup X_n \ .$$

which is ii) of $\mathbf{Claim}_{r+1}$.

We now prove iii) of $\mathbf{Claim}_{r+1}$. By Bézout's theorem, by the definition of X'_{r+1} and by $\deg Z_i \leq \theta$ we deduce

$$\deg X'_{r+1} \leq \theta \Big(\sum_{i=1}^{s} \deg W_i \Big) + \deg X_{r+1} \ .$$

Substituting $\sum_{i=1}^{s} \deg W_i = \deg X'_r - \deg G_r$ (which rises directly from (7.2)), we obtain

$$\deg X'_{r+1} \leq \theta(\deg X'_r - \deg G_r) + \deg X_{r+1} \ .$$

Thus

$$\begin{aligned} \sum_{i=k}^{r} \theta^{r+1-i} \deg G_i + \deg X'_{r+1} &\leq \sum_{i=k}^{r} \theta^{r+1-i} \deg G_i \\ &\qquad + \theta(\deg X'_r - \deg G_r) + \deg X_{r+1} \\ &= \theta \Big(\sum_{i=k}^{r-1} \theta^{r-i} \deg G_i + \deg X'_r \Big) + \deg X_{r+1} \ . \end{aligned}$$

By iii) of $\mathbf{Claim}_r$ we have

$$\theta\Big(\sum_{i=k}^{r-1}\theta^{r-i}\deg G_i+\deg X_r'\Big)+\deg X_{r+1}\leq\sum_{i=k}^{r+1}\theta^{r+1-i}\deg X_i\ .$$

This proves iii) of $\mathbf{Claim}_{r+1}$.

Exercice 7.1. *Let V be an irreducible variety of codimension k which is not a translate of a subtorus and $B\subseteq V$ a translate of a subtorus of relative codimension 1. Let $\theta=\delta(V)\big(200n^5\log(n^2\delta(V))\big)^{2(k+1)n}$. Use theorem 7.1 to show that if $\delta_0(B)>\theta$ then $B(\theta^{-1})$ is empty.*

Exercice 7.2. *Let $V\subseteq W$ be subvarieties of $\mathbb{G}_{\mathrm{m}}^n$. Let*

$$\theta=\delta(W)\big(200n^5\log(n^2\delta(W))\big)^{(n-k)n(n-1)}\ .$$

Use theorem 7.3 to show that $V(\theta^{-1})\subseteq\bigcup B_j$ where the $B_j\subseteq W$ are translates of tori such that $\delta_0(B_j)\leq\theta$ and $\sum_j\deg(B_j)\leq\theta^n$.

Exercice 7.3. *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$ which is not a translate of a subtorus. Let $\theta_0=\delta_0(V)\big(200n^5\log(n^2\delta_0(V))\big)^{n(n-1)^2}$. Use exercise 7.2 to show that $V(\theta_0^{-1})$ is contained in a finite union of translates B_j of proper subtori such that $V\not\subseteq B_j$, $\delta_0(B_j)\leq\theta_0$ and $\sum_j\deg(B_j)\leq\theta_0^n$.*

Exercice 7.4. *Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Let*

$$\theta_\omega=\omega(V)\big(200n^5\log(n^2\omega(V))\big)^{n(n-1)^2}\ .$$

Use exercise 7.2 to show that $V(\theta_\omega^{-1})$ is contained in a finite union of translates B_j of proper subtori such that $\delta_0(B_j)\leq\theta_\omega$ and $\sum_j\deg(B_j)\leq\theta_\omega^n$. Note that this implies that, for V transverse, $\hat{\mu}^{\mathsf{ess}}(V)\geq\theta_\omega^{-1}$.

Exercice 7.5. *Let V be a $\mathbb{Q}$-irreducible weak transverse subvariety of $\mathbb{G}_{\mathrm{m}}^n$. Deduce a lower bound for $\hat{\mu}^{\mathsf{ess}}(V)$ depending on $\omega(V)$ from theorem 7.2.*

Exercice 7.6. *Let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^n$ $\mathbb{Q}$-irreducible weak transverse . Deduce a lower bound for $\mu^*(V)$ depending on $\delta(V)$ from theorem 7.2.*

Exercice 7.7. *Let V be a subvariety of $\subseteq \mathbb{G}_{\mathrm{m}}^n$ of codimension k and let $B_1, \ldots, B_t$ be the maximal torsion subvarieties of V (see exercise 1.2). Define $\theta = \delta(V)\big(200n^5 \log(n^2\delta(V))\big)^{(n-k)n(n-1)}$. Show that $\delta_0(B_j) \leq \theta(V)$ and*

$$\sum_{j=1}^{t} \theta(V)^{\dim(B_j)} \deg(B_j) \leq \theta^n$$

(and thus $t \leq \theta^n$).

Exercice 7.8. *Prove remark 7.4.*

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