Appendix (by F. Amoroso)

Lower bounds for the height

VI.1 Introduction

The former Manin-Mumford conjecture predicted that the set of torsion points of a curve of genus ≥ 2 embedded in its jacobian is finite. More generally, let $\mathbb{G}$ be a semi-abelian variety and let V be an irreducible[1] algebraic subvariety of $\mathbb{G}$, defined over some algebraically closed field K. We say that V is a *torsion variety* if V is a translate of a proper subtorus by a torsion point of $\mathbb{G}$. We also denote by V_{tors} the set of torsion points of $\mathbb{G}$ lying on V. Then we have the following generalization of the Manin-Mumford conjecture.

Theorem VI.1.1

i) If V is not a torsion variety, then the set V_{tors} of torsion points of $\mathbb{G}$ lying on V is not Zariski dense.

ii) The Zariski closure of V_{tors} is a finite union of torsion varieties.

The two assertions are clearly equivalent. Theorem VI.1.1 was proved by Raynaud [31] when $\mathbb{G}$ is an abelian variety, by Laurent [26] if $\mathbb{G} = \mathbb{G}_{\mathrm{m}}^n$, and finally by Hindry [24] in the general situation.

We assume from now on that all varieties are algebraic and defined over $\overline{\mathbb{Q}}$. Bogomolov [13] gave the following generalization of the former Manin-Mumford conjecture. Let $\mathcal{C}$ be a curve of genus ≥ 2 embedded in its jacobian. Then $\mathcal{C}(\overline{\mathbb{Q}})$ is discrete for the metric induced by the Néron-Tate height. In other words, Bogomolov conjectures that the set of points of "sufficiently small" height on $\mathcal{C}$ is finite, while the former Manin-Mumford conjecture makes a similar assertion on the set of torsion points (which are precisely the points of height zero).

More generally, let $\mathbb{G}$ be a semi-abelian variety and let $\hat{h}$ be a normalized height on $\mathbb{G}(\overline{\mathbb{Q}})$. Hence, $\hat{h}$ is the Neron-Tate height if $\mathbb{G}$ is abelian, and it is the Weil height if $\mathbb{G} = \mathbb{G}_{\mathrm{m}}^n \hookrightarrow \mathbb{P}^n$. In particular, $\hat{h}$ is a non-negative function on $\mathbb{G}$, and $\hat{h}(P) = 0$ if and only if P is a torsion point. Given an algebraic subvariety of $\mathbb{G}$, we denote by V^* the complement in V of the Zariski closure of the set of torsion points of V. Therefore, by theorem VI.1.1, $V \backslash V^* = \overline{V_{\mathrm{tors}}}$ is a finite union of torsion varieties.

Theorem VI.1.2 *Let V be an irreducible subvariety of a semi-abelian variety $\mathbb{G}$. Then:*

i) If V is not a torsion variety, then there exists $\theta > 0$ such that the set $V(\theta) = \{P \in V \text{ s.t. } \hat{h}(P) \leq \theta\}$ is not Zariski dense in V.

ii) V^* is discrete for the metric induced by $\hat{h}$, i.e.

$$\inf\{\hat{h}(P) \text{ s.t. } P \in V^*\} > 0.$$

[1] By irreducible we mean geometrically irreducible

It is easy to see that the two assertions are equivalent. In this formulation, theorem VI.1.2 was proved for $\mathbb{G} = \mathbb{G}_{\mathrm{m}}^n$ by Zhang (see [37]). In the abelian case, Ullmo (see [35]) proved Bogomolov's original formulation for curves ($\dim(V) = 1$); immediately after Zhang (see [38]) proved theorem VI.1.2. The semi-abelian case was solved by David and Philippon (see [21]).

In this appendix we shall describe some quantitative versions of theorem VI.1.2 for a torus $\mathbb{G} = \mathbb{G}_{\mathrm{m}}^n$, and we sketch proofs of theorems which prove these conjectures "up to an ε".

VI.2 Algebraic numbers.

In this section we first recall some facts from sections 2.1, 2.2 and 2.3, of Chapter III.

Let $\alpha \in \overline{\mathbb{Q}}$ and let K be *any* number field containing α. We denote by $\mathcal{M}_K$ the set of places of K. For $v \in K$, let K_v be the completion of K at v and let $|\cdot|_v$ be the (normalized) absolute value of the place v. Hence

$$|\alpha|_v = |\sigma\alpha|,$$

if v is an archimedean place associated to the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. Note that two conjugate embeddings define the same place. If v is a non archimedean place associated with the prime ideal $\wp$ over the rational prime p, we have

$$|\alpha|_v = p^{-\lambda/e},$$

where e is the ramification index of $\wp$ and λ is the exponent of $\wp$ in the factorization of the ideal (α) in the ring of integers of K. Thus $\|\alpha\|_v = |\alpha|_v^{[K_v:\mathbb{Q}_v]}$, in the notation of Chapter III, section 2.1. Our normalization agrees with the product formula

$$\prod_{v \in \mathcal{M}_K} |\alpha|_v^{[K_v:\mathbb{Q}_v]} = 1$$

which holds for any $\alpha \in K^*$.

For further reference, we recall that for any rational place w (thus $w = \infty$ or $w =$ a prime number),

$$\sum_{v|w} [K_v : \mathbb{Q}_v] = [K : \mathbb{Q}].$$

We define the Weil height of α by

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log \max\{|\alpha|_v, 1\}.$$

It is easy to see that this definition does not depend on the field K containing α; it thus defines a function $h\colon \overline{\mathbb{Q}} \to \mathbb{R}^+$.

The Weil height of an algebraic number is related to the Mahler measure of a polynomial. Let $P \in \mathbb{C}[x]$ be non-zero; then its Mahler measure is

$$M(P) = \exp \int_0^1 \log \left|P\left(e^{2\pi i t}\right)\right| dt.$$

We also agree that $M(0) = 0$. The Mahler measure has some nice properties. It is a multiplicative function, and it is invariant by the morphism $P(x) \to P(x^l)$ ($l \in \mathbb{N}$). Let $\alpha_1, \ldots, \alpha_d$ be the roots of P and let P_d be its leading coefficient. By Proposition III.2.5,

$$M(P) = |P_d| \prod_{j=1}^{d} \max\{|\alpha_j|, 1\}. \qquad \text{(VI.2.1)}$$

Let K be a number field, and let $f \in K[\mathbf{x}]$. We define:

$$\hat{h}(f) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log M_v(f),$$

where $M_v(f)$ is the maximum of the v-adic absolute values of the coefficients of f if v is non archimedean, and $M_v(f)$ is the Mahler measure of σf if v is an archimedean place associated with the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. As for the Weil height, this definition does not depend on the field K containing the coefficients of f. Moreover, by the product formula, $\hat{h}(\lambda f) = \hat{h}(f)$ for any $\lambda \in K^*$. We also remark that $\hat{h}$ is an additive function. Indeed, $M_v(*)$ is a multiplicative function at least for $v \mid \infty$. By a simple exercise this property still holds for $v \nmid \infty$. By the above properties and by (VI.2.1), $\hat{h}(f)$ is the sum of the Weil height of its roots. As a special case

$$h(\alpha) = \frac{\log M(f)}{[\mathbb{Q}(\alpha):\mathbb{Q}]}. \tag{VI.2.2}$$

where $f \in \mathbb{Z}[x]$ is the minimal polynomial of α over $\mathbb{Z}$ (*i.e.* f is irreducible in $\mathbb{Z}[x]$, $f(\alpha) = 0$ and its leading coefficient is positive).

Let $\|P\|_1$ be the sum of the absolute values of the coefficients of $P \in \mathbb{C}[x]$ (the "length" of P). Since the maximum of $|P|$ on the unit disk is bounded by $\|P\|_1$, we have $M(P) \leq \|P\|_1$. Moreover,

$$\|P\|_1 \leq 2^{\deg(P)} M(P). \tag{VI.2.3}$$

This follows from (VI.2.1) and from the usual formulas for the coefficients of a polynomial as symmetric functions of its roots. Inequality (VI.2.3) implies a theorem of Northcott: the set of algebraic numbers of bounded height and degree is finite. If $h(\alpha) \leq B$, by the above inequality the coefficients of the minimal polynomial of α are bounded by $2^{[\mathbb{Q}(\alpha):\mathbb{Q}]}B$. Thus the minimal polynomials of the algebraic numbers of bounded height and degree belong to a finite set.

We now state some other important properties of the height. Let α, $\beta \in \overline{\mathbb{Q}}^*$. Then $h(\alpha\beta) \leq h(\alpha) + h(\beta)$. This follows from the inequality $\max\{xy, 1\} \leq \max\{x, 1\}\max\{y, 1\}$ (for x, $y > 0$) applied at each place. Moreover, if β is a root of unity, $h(\alpha\beta) = h(\alpha)$. Indeed roots of unity have absolute value 1 at each place. Let $\alpha \in \overline{\mathbb{Q}}$ and $n \in \mathbb{Z}$. Then $h(\alpha^n) = |n|h(\alpha)$. If $n \geq 0$, this is obvious from the definition, while, if $n < 0$, this follows from the fact that $h(\alpha^{-1}) = h(\alpha)$, by the product formula.

This last property implies that $h(\alpha) = 0$ if and only if α is a root of unity. This is a theorem of Kronecker, and it is precisely the simplest case of Zhang's theorem on the Bogomolov toric conjecture. The problem of finding sharp lower bounds for the height of a non-zero algebraic number α which is not a root of unity is a famous problem of Lehmer. Let $f \in \mathbb{Z}[x]$ be a nonconstant irreducible polynomial. Assume that $f \neq \pm x$ and that $\pm f$ is not a cyclotomic polynomial. Lehmer (see [27]) asks whether there exists an absolute constant $C > 1$ such that $M(f) \geq C$. An equivalent formulation in terms of the height is the following. Let α be a non-zero algebraic number of degree d which is not a root of unity. Then *Lehmer's conjecture* may be stated as follows: there exists an absolute constant $c > 0$ such that

$$h(\alpha) \geq \frac{c}{d}.$$

This should be the best possible lower bound for the height (without any further assumption on α), since $h(2^{1/d}) = (\log 2)/d$. The best known result in the direction of Lehmer's conjecture is the following theorem.

Theorem VI.2.1 (Dobrowolski, 1979) *For any algebraic number $\alpha \in \overline{\mathbb{Q}}^*$ of degree $d \geq 2$ which is not a root of unity we have*

$$h(\alpha) \geq \frac{c}{d}\left(\frac{\log d}{\log\log d}\right)^{-3}$$

for some absolute constant $c > 0$.

In the original statement [22] $c = 1/1200$; later Voutier [36] shows that one can take $c = 1/4$.

VI.2.1 Sketch of the proof of theorem VI.2.1

We may assume that α is an algebraic integer, otherwise $h(\alpha) \geq (\log 2)/d$. Let f be its minimal polynomial over $\mathbb{Z}$ and let p be a prime number. Then, by Fermat's little theorem,

$$f(x)^p \equiv f(x^p) \bmod p\mathbb{Z}[x].$$

Thus

$$|f(\alpha^p)|_v \leq p^{-1}$$

for any $v \mid p$. Let $F \in \mathbb{Z}[x]$ be a polynomial of degree L vanishing on α with multiplicity $\geq T$ for some *parameters* L and T with $L \geq dT$. Then

$$|F(\alpha^p)|_v \leq p^{-T}$$

for any $v \mid p$. Moreover $|F(\alpha^p)|_v \leq 1$ for $v \nmid \infty$ and

$$|F(\alpha^p)|_v \leq \|F\|_1 \max(1, |\alpha|_v)^{pL}$$

if $v \mid \infty$. Assume that

$$F(\alpha^p) \neq 0. \tag{VI.2.4}$$

Then, by the product formula,

$$\begin{aligned} 0 &= \sum_v \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |F(\alpha^p)|_v \\ &\leq \sum_{v|p} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |F(\alpha^p)|_v + \sum_{v|\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log |F(\alpha^p)|_v \\ &\leq -\sum_{v|p} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} T \log p + \sum_{v|\infty} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} (\log \|F\|_1 + pL \log^+ |\alpha|_v) \\ &\leq -T \log p + \log \|F\|_1 + pLh(\alpha). \end{aligned}$$

This yields

$$h(\alpha) \geq \frac{T \log p - \log \|F\|_1}{pL}. \tag{VI.2.5}$$

We choose $L = d$, $T = 1$ and $F = f$. The non vanishing condition (VI.2.4) is satisfied. Indeed, if α is not a root of unity, then α^p is not a conjugate of α, otherwise $ph(\alpha) = h(\alpha^p) = h(\alpha)$ and α would be a root of unity. Thus we obtain

$$h(\alpha) \geq \frac{\log p - \log \|f\|_1}{pd} .$$

Unfortunately, $\log \|f\|_1$ can be as large as a power of d, even if the height of α is very small (see [1]). Thus, to get a positive lower bound, we must choose p to be exponential in d^c, and the argument terminates with a poor lower bound of the shape $h(\alpha) \geq e^{-d^c}$.

The use of Siegel's Lemma [15], a classical tool in diophantine approximation, improves enormously the quality of this bound. Using this lemma, we find a non-zero polynomial $F \in \mathbb{Z}[x]$ ("auxiliary function") of degree $\leq L$ vanishing on α with multiplicity $\geq T$ as required and such that

$$\log \|F\|_\infty \leq \frac{dT}{L + 1 - dT} (T \log(L + 1) + Lh(\alpha)). \tag{VI.2.6}$$

Here $\|F\|_\infty$ denotes the maximum of the absolute values of the coefficients of F.

The proof now follows the scheme of a classical transcendence proof: choice of the auxiliary function, extrapolation and zero's lemma. During the proof we assume that the height of α is pathologically small and we argue for a contradiction.

Let A and B positive real functions of d. We write $A \ll B$ if and only if $A \leq cB$ for some $c > 0$, and $A \approx B$ if both $A \ll B$ and $B \ll A$. We shall also denote by $c_1, \ldots, c_4$ positive constants.

• *Choice of the auxiliary function.* Since $\log \|F\|_1 \leq (L+1) \log \|F\|_\infty$, by (VI.2.6) we have

$$\log \|F\|_1 \leq \log(L+1) + \frac{dT}{L+1-dT}\left(T\log(L+1) + Lh(\alpha)\right).$$

This inequality cannot give anything better than $\log \|F\|_1 \ll \log(L+1)$. Therefore, it is reasonable to choose L and T in such a way that

$$\frac{dT^2}{L+1-dT} \approx 1,$$

say $L = dT^2$, and to assume $Lh(\alpha) \leq T\log(L+1)$. Under these assumptions, the length of the auxiliary polynomial satisfies

$$\log \|F\|_1 \ll \log(L+1) \leq \log(dT^2).$$

• *Zero's lemma.* In order to apply (VI.2.5), we need a prime p such that $F(\alpha^p) \neq 0$. We fix a third parameter N and we assume $F(\alpha^p) = 0$ for any prime p with $N \leq p \leq 2N$. Since $F \in \mathbb{Z}[x]$, then F vanishes also in the conjugates of α^p. Since α is not a root of unity, α^{p_1} and α^{p_2} are not conjugate for primes $p_1 \neq p_2$, since otherwise $p_1 h(\alpha) = p_2 h(\alpha)$ and α would be a root of unity. Assume

$$[\mathbb{Q}(\alpha^n) : \mathbb{Q}] = d \tag{VI.2.7}$$

for all integer n. Then the set of the conjugate of α^p with p prime, $N \leq p \leq 2N$, has cardinality $d\pi(N)$ with π is as usual the prime-counting function. By Chebyshev Theorem, $\pi(N) \geq c_1 N/\log N$. Since a polynomial as at most as many zero as its degree,

$$L \leq \frac{c_1 dN}{\log N}.$$

We choose the smallest N such that this equality is not satisfied. Thus $N \approx \frac{L}{d}\log(L/d)$. By the choice $L = dT^2$,

$$N \approx T^2 \log T.$$

• *Conclusion.* By the remarks above, there exists a prime $p \in [N, 2N]$ such that $F(\alpha^p) \neq 0$. By (VI.2.5), by our choices $L = dT^2$, $N \approx T^2 \log T$, and by the bound $\log \|F\|_1 \ll \log(L+1) \leq \log(dT^2)$ on the length of the auxiliary polynomial,

$$h(\alpha) \geq \frac{c_2 T \log T - c_3 \log(dT^2)}{dT^4 \log T}.$$

Again, this inequality cannot give anything better than $h(\alpha) \geq \frac{c_2 T \log T}{dT^4 \log T} = \frac{c_2}{dT^3}$. It is thus reasonable to choose the smallest T such that $c_2 T \log T \geq 2c_3 \log(dT^2)$, that is

$$T \approx \frac{\log d}{\log\log d}.$$

This gives

$$h(\alpha) \geq \frac{c_2}{2dT^3} \geq \frac{c_4}{d}\left(\frac{\log d}{\log\log d}\right)^{-3}.$$

Note that if our working assumption $Lh(\alpha) \leq T\log(L+1)$ is not satisfied, we get the bound:

$$h(\alpha) \geq \frac{T}{L}\log(L+1) = \frac{\log(dT^2+1)}{dT} \approx \frac{\log\log d}{d}$$

which is even better than Lehmer!

Dobrowolski's theorem is proved under the additional assumption (VI.2.7). In the general case we proceed by induction on d. It is useful to replace the reminder term by the decreasing function

$$d \mapsto \varepsilon(d) := \left(\frac{\log 5d}{\log\log 3d}\right)^{-3}.$$

Let α be an algebraic number of degree $d \geq 1$ and assume

$$d'h(\beta) \geq \varepsilon(d')$$

for all algebraic numbers $\beta \in \overline{\mathbb{Q}}^*$ different from a root of unity and with $d' = [\mathbb{Q}(\beta) : \mathbb{Q}] < d$. From the first part of the proof (and if c is sufficiently large), we can assume that for some $n > 1$ (VI.2.7) does not hold. We follow an argument of [32]. We have $k = [\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha^n)] > 1$. Let β be the norm of α from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}(\alpha^n)$. Then $\beta = \zeta\alpha^k$ for some root of unity ζ and $h(\beta) = h(\alpha^k) = kh(\alpha)$. Since $d' = [\mathbb{Q}(\beta) : \mathbb{Q}] < d$ and since $t \mapsto \varepsilon(t)$ decreases,

$$dh(\alpha) = [\mathbb{Q}(\alpha^n) : \mathbb{Q}]h(\beta) \geq d'h(\beta) \geq c\varepsilon(d') \geq c\varepsilon(d).$$

$\square$

VI.2.2 Height in abelian extensions

In some special cases, not only Lehmer's conjecture is true, but it can also be sharpened. Assume for instance that L is a totally real number field or a CM field (a totally complex quadratic extension of a totally real number field). Then, as a special case of a more general result, Schinzel proved that

$$h(\alpha) \geq \frac{1}{2}\log\frac{1+\sqrt{5}}{2} = 0.2406...$$

if $\alpha \in L^*$ *and* $|\alpha| \neq 1$. In particular, by Kronecker's theorem, this inequality holds if α is an algebraic integer different from zero and from a root of unity. It may happen that algebraic numbers of absolute value 1 in CM fields have arbitrary small Weil height. Let for instance $\alpha = (\sqrt{2} - i)/(\sqrt{2} + i)$. Then all the algebraic conjugates of α have absolute value 1. Thus, the same property holds for the algebraic conjugates of $\alpha^{1/d}$, where d is an arbitrary positive integer. In turns, this implies that $\mathbb{Q}(\alpha^{1/d})$ is a CM field. Nevertheless, we have $0 < h(\alpha^{1/d}) = h(\alpha)/d \to 0$ as $d \mapsto \infty$.

When the extension $L/\mathbb{Q}$ is an imaginary Galois extension, L is CM if and only if the complex conjugation lies in the center of the Galois group. Assume further that $L/\mathbb{Q}$ is abelian. In [8] we prove

Theorem VI.2.2 (A. – Dvornicich, 2000) *Let $L/\mathbb{Q}$ be an abelian extension, and let $\alpha \in L^*$, α not a root of unity. Then*

$$h(\alpha) \geq \frac{\log 5}{12} = 0.1341...$$

The above lower bound is not far from the best possible one. Let L be the 21-th cyclotomic field. We recall that L is one of the 29 cyclotomic fields with class number one. The prime 7 splits as $(P\overline{P})^6$ in the ring of integer of L and P is a prime ideal of norm 7. Let γ be a generator of P and define $\alpha = \gamma/\overline{\gamma}$. Then

$$|\alpha|_v^{[L_v:\mathbb{Q}_v]} = \begin{cases} 7^{-1}, & \text{if } v \text{ is over } P; \\ 7, & \text{if } v \text{ is over } \overline{P}; \\ 1, & \text{otherwise.} \end{cases}$$

Thus

$$h(\alpha) = \frac{\log 7}{12}.$$

This example shows that numbers of small height in an abelian extension are closely related to the class number problem. We can reverse the above construction and use lower bounds for the height to obtain informations on the size of the ideal class group of some fields. For instance, let L_m be the m-th cyclotomic field, and define e_m to be the exponent of its class group, *i.e.* the smallest positive integer e such that I^e is a principal ideal for all integral ideals I of L_m. By Linnik's theorem, there exists an absolute constant $c > 0$ and a prime $p \leq m^c$ which splits completely in L_m. Let P be a prime ideal of L_m over p; by definition $P^{e_m} = (\gamma)$ for some integer $\gamma \in L_m$. Define $\alpha = \gamma/\overline{\gamma}$. The above argument shows that

$$h(\alpha) = \frac{e_m \log p}{[L_m : \mathbb{Q}]} \leq \frac{e_m c \log m}{\varphi(m)} \ ,$$

where $\varphi(\cdot)$ is the Euler function. Since $L_m/\mathbb{Q}$ is abelian,

$$\frac{\log 5}{12} \leq h(\alpha) \leq \frac{e_m c \log m}{\varphi(m)} \ .$$

We obtain:

$$e_m \geq \frac{\log 5}{12c} \times \frac{\varphi(m)}{\log m} \ .$$

Let K be a CM field of discriminant Δ and degree d. We assume the Generalized Riemann Hypothesis for the Dedekind zeta function of K. More sophisticated argument show (see [9]) that for any $\varepsilon > 0$ the exponent e_K of the class group of K satisfies:

$$e_K \geq \max\left\{ \frac{C \log|\Delta|}{d \log\log|\Delta|}, C(\varepsilon) d^{1-\varepsilon} \right\} \ ,$$

where C and $C(\varepsilon)$ are positive constants. Thus the exponent of the class group of a CM field goes to infinity with its discriminant.

We can "mix" the lower bound in abelian extensions (theorem VI.2.2) with Dobrowolski's result, theorem VI.2.1. Let K be a fixed number field, and let L/K be an abelian extension. In [11], we prove that for $\alpha \in L^*$ not a root of unity,

$$h(\alpha) \geq \frac{c(K)}{D} \left(\frac{\log 2D}{\log\log 5D} \right)^{-13} \ , \tag{VI.2.8}$$

where $D = [L(\alpha) : L]$ and where $c(K) > 0$. In the proof of [11], $c(K)$ depended on both the degree *and* the discriminant of K.

We come back to the lower bounds for the height on an abelian extension L of a number field K. As a very special case of (VI.2.8), the height in L^*, outside the set of roots of unity, is bounded from below by a positive function depending only on K. The following question arises: is it true that we can choose a function depending only on the degree $[K : \mathbb{Q}]$? In [12] we gave a positive answer to this problem. Let L/K be as before. Then for any $\alpha \in L^*$ which is not a root of unity, we have

$$h(\alpha) > 3^{-d^2-2d-6}$$

where $d = [K : \mathbb{Q}]$. This result has some amusing consequences. For instance, let L be a dihedral extension of the rational field of degree $2n$, say. Then L is an abelian extension of its quadratic subfield K fixed by the normal cyclic group of order n. Thus for any $\alpha \in L^*$ which is not a root of unity we have

$$h(\alpha) \geq 3^{-14}.$$

VI.2.3 Sketch of the proof of theorem VI.2.2

For a natural number $m \geq 3$ we denote by ζ_m a primitive m th-root of unity, and we let $L_m = \mathbb{Q}(\zeta_m)$ be the m-th cyclotomic field. We need two lemmas. Let $p \geq 3$ be a prime number, and let $\alpha \in L_m^*$, α not a root of unity. We show that

$$h(\alpha) \geq \frac{\log(p/2)}{2p} \ .$$

Choosing $p = 5$, this gives, *via* Kronecker-Weber's theorem, the lower bound

$$h(\alpha) \geq \frac{\log(5/2)}{10}$$

for the height of a non-zero algebraic number α (α not a root of unity) lying in an abelian extension. A refinement of the proof gives the more precise result of theorem VI.2.2.

The following simple lemma is the key argument in the proof.

Lemma VI.2.3 *Let p be a rational prime. Then there exists $\sigma = \sigma_p \in \mathrm{Gal}(L_m/\mathbb{Q})$ with the following two properties.*

i) If $p \nmid m$, then

$$p \mid (\gamma^p - \sigma\gamma)$$

for any integer $\gamma \in L_m$.

ii) If $p \mid m$, then

$$p \mid (\gamma^p - \sigma\gamma^p)$$

for any integer $\gamma \in L_m$. Moreover, if $\sigma\gamma^p = \gamma^p$ for some $\gamma \in L_m$, then there exists a root of unity $\zeta \in L_m$ such that $\zeta\gamma$ is contained in a proper cyclotomic subextension of L_m.

Proof. Assume first that $p \nmid m$. Let $\sigma \in \mathrm{Gal}(L_m/\mathbb{Q})$ be the Frobenius automorphism defined by $\sigma\zeta_m = \zeta_m^p$. For any integer $\gamma \in L_m$, we have $\gamma = f(\zeta_m)$ for some $f \in \mathbb{Z}[x]$. Hence

$$\gamma^p \equiv f(\zeta_m^p) \equiv f(\sigma\zeta_m) \equiv \sigma\gamma \pmod p.$$

Assume now that $p|m$. The Galois group $\mathrm{Gal}(L_m/K_{m/p})$ is cyclic of order $k = p$ or $k = p-1$ depending on whether $p^2|m$ or not. Let σ be one of its generators; hence $\sigma\zeta_m = \zeta_p\zeta_m$ for some primitive p-th root of unity ζ_p. For any integer $\gamma = f(\zeta_m) \in \mathbb{Z}[\zeta_m]$, we have

$$\gamma^p \equiv f(\zeta_m^p) \equiv f(\sigma\zeta_m^p) \equiv \sigma\gamma^p \pmod p.$$

Suppose finally that $\sigma\gamma^p = \gamma^p$: then $\sigma\gamma = \zeta_p^u\gamma$ for some integer u. It follows that $\sigma(\gamma/\zeta_m^u) = \gamma/\zeta_m^u$, hence γ/ζ_m^u belongs to the fixed field $K_{m/p}$, as desired.

□

Let $L = L_m$, and let $\sigma = \sigma_p$ be the homomorphism given by lemma VI.2.3. Assume first that $p \nmid m$. Let v be a place of L dividing p (thus $|p|_v = 1/p$). By the "strong approximation theorem" (see for instance [17], Chapter II, § 15, page 67), we see easily that there exists an algebraic integer $\beta = \beta_v \in L$ such that $\alpha\beta$ is integer and

$$|\beta|_v = \max\{1, |\alpha|_v\}^{-1}.$$

Then

$$|(\alpha\beta)^p - \sigma(\alpha\beta)|_v \leq p^{-1} \quad \text{and} \quad |\beta^p - \sigma\beta|_v \leq p^{-1}.$$

Using the ultrametric inequality, we deduce that

$$\begin{aligned} |\alpha^p - \sigma\alpha|_v &= |\beta|_v^{-p} |(\alpha\beta)^p - \sigma(\alpha\beta) + (\sigma\beta - \beta^p)\sigma\alpha|_v \\ &\le |\beta|_v^{-p} \max\left(|(\alpha\beta)^p - \sigma(\alpha\beta)|_v, |\beta^p - \sigma\beta|_v |\sigma\alpha|_v\right) \\ &\le p^{-1} \max(1, |\alpha|_v)^p \max(1, |\sigma\alpha|_v). \end{aligned}$$

Suppose now that v is a finite place not dividing p. Then we have

$$|\alpha^p - \sigma(\alpha)|_v \le \max(1, |\alpha|_v)^p \max(1, |\sigma(\alpha)|_v) \ .$$

Finally, if $v|\infty$,

$$|\alpha^p - \sigma(\alpha)|_v \le 2 \max(1, |\alpha|_v)^p \max(1, |\sigma(\alpha)|_v) \ .$$

Moreover $\alpha^p \neq \sigma\alpha$, since α is not a root of unity. We now apply the product formula to $\gamma = \alpha^p - \sigma\alpha$, using

$$\sum_{v|p} [L_v : \mathbb{Q}_v] = \sum_{v|\infty} [L_v : \mathbb{Q}_v] = [L : \mathbb{Q}].$$

We get

$$\begin{aligned} 0 &= \sum_{\substack{v\nmid\infty \\ v\nmid p}} \frac{[L_v : \mathbb{Q}_v]}{[L : \mathbb{Q}]} \log|\gamma|_v + \sum_{v|p} \frac{[L_v : \mathbb{Q}_v]}{[L : \mathbb{Q}]} \log|\gamma|_v + \sum_{v|\infty} \frac{[L_v : \mathbb{Q}_v]}{[L : \mathbb{Q}]} \log|\gamma|_v \\ &\le \sum_{v} \frac{[L_v : \mathbb{Q}_v]}{[L : \mathbb{Q}]} (p \log^+ |\alpha|_v + \log^+ |\sigma\alpha|_v) - \sum_{v|p} \frac{[L_v : \mathbb{Q}_v]}{[L : \mathbb{Q}]} \log p + \sum_{v|\infty} \frac{[L_v : \mathbb{Q}_v]}{[L : \mathbb{Q}]} \log 2 \\ &= ph(\alpha) + h(\sigma\alpha) - \log p + \log 2 \\ &= (p+1)h(\alpha) - \log(p/2). \end{aligned}$$

Therefore,

$$h(\alpha) \ge \frac{\log(p/2)}{p+1} \ge \frac{\log(p/2)}{2p}.$$

Assume now that $p \mid m$. Let v be a place of L dividing p and let $\beta = \beta_v \in L$ as in the first part of the proof. Then

$$|(\alpha\beta)^p - \sigma(\alpha\beta)^p|_v \le p^{-1} \quad \text{and} \quad |\beta^p - \sigma\beta^p|_v \le p^{-1}.$$

Using the ultrametric inequality, we find

$$\begin{aligned} |\alpha^p - \sigma\alpha^p|_v &= |\beta|_v^{-p} |(\alpha\beta)^p - \sigma(\alpha\beta)^p + (\sigma\beta^p - \beta^p)\sigma\alpha^p|_v \\ &\le p^{-1} \max(1, |\alpha|_v)^p \max(1, |\sigma\alpha|_v)^p. \end{aligned}$$

Moreover, we can assume $\alpha^p \neq \sigma\alpha^p$. Otherwise, by lemma VI.2.3, there would exist a root of unity $\zeta \in L$ such that $\zeta\alpha$ is contained in a proper cyclotomic subextension of L; hence $h(\alpha) = h(\zeta\alpha)$ and, by induction, $h(\zeta\alpha) \ge \frac{\log(p/2)}{2p}$. Applying the product formula to $\gamma = \alpha^p - \sigma\alpha^p$ as in the first part of the proof, we get

$$0 \le ph(\alpha) + ph(\sigma\alpha) - \log p + \log 2 = 2ph(\alpha) - \log(p/2).$$

Again

$$h(\alpha) \ge \frac{\log(p/2)}{2p}.$$

□

VI.3 Subvarieties of $\mathbb{G}_{\mathrm{m}}^n$.

We consider a torus $\mathbb{G}_{\mathrm{m}}^n$, and we fix the "standard embedding" $\iota\colon \mathbb{G}_{\mathrm{m}}^n \hookrightarrow \mathbb{P}^n$,

$$\iota(x_1, \ldots, x_n) = (1 : x_1 : \cdots : x_n).$$

By a subvariety of $\mathbb{G}_{\mathrm{m}}^n$ we mean an algebraic subvariety V defined over some number field K. The degree of V is the degree of its Zariski closure in $\mathbb{P}^n$. We shall say that V is irreducible if its Zariski closure is geometrically irreducible. Similarly, we say that V is irreducible over K if its Zariski closure is irreducible over K.

We recall some definitions from chapter IV, section 2.2. Given $\boldsymbol{\lambda} \in \mathbb{Z}^n$ and $\mathbf{x} = (x_1, \ldots, x_n)$ we set $\mathbf{x}^{\boldsymbol{\lambda}} = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. Given any m-tuple of vectors $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m \in \mathbb{Z}^n$ we define a regular map $\varphi\colon \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^m$ by $\varphi(\mathbf{x}) := (\mathbf{x}^{\boldsymbol{\lambda}_1}, \ldots, \mathbf{x}^{\boldsymbol{\lambda}_m})$. This map is an algebraic group homomorphism, called *monoidal*. When $m = n$, the homomorphism φ is invertible if and only if $\det(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m) = \pm 1$; in this case it is called a *monoidal automorphism* of $\mathbb{G}_{\mathrm{m}}^n$. If $\det(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_m) \neq 0$ the kernel of φ is finite; we shall say that φ is finite. We shall often use a special finite monoidal morphism. Let $l \in \mathbb{N}$. We denote by $[l]\colon \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^n$ the "multiplication" by $[l]$, *i.e.* the morphism $\mathbf{x} \mapsto \mathbf{x}^l = (x_1^l, \ldots, x_n^l)$. Thus the kernel $\mathrm{Ker}[l]$ is the set of l-torsion points. It is a subgroup isomorphic to $(\mathbb{Z}/l\mathbb{Z})^n$.

By algebraic subgroup of $\mathbb{G}_{\mathrm{m}}^n$ we mean a closed algebraic subvariety stable under the group operations. An irreducible algebraic subgroup is called a torus. Any algebraic subgroup is a finite disjoint union of translates of a torus. Given an algebraic subgroup H we denote by H^0 its connected component containing the neutral element. Let $\Lambda \subseteq \mathbb{Z}^n$ be a subgroup. Then

$$H_\Lambda = \{\mathbf{x} \in \mathbb{G}_{\mathrm{m}}^n,\ \forall \boldsymbol{\lambda} \in \Lambda,\ \mathbf{x}^{\boldsymbol{\lambda}} = 1\}$$

is an algebraic group. Moreover, $\Lambda \mapsto H_\Lambda$ is a bijection between subgroups of $\mathbb{Z}^n$ and algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^n$.

Let V be an irreducible subvariety of $\mathbb{G}_{\mathrm{m}}^n$. We define its stabilizer to be

$$\mathrm{Stab}(V) = \{\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n \text{ s.t. } \boldsymbol{\alpha} V = V\}.$$

Thus

$$\mathrm{Stab}(V) = \bigcap_{\mathbf{x} \in V} \mathbf{x}^{-1} V.$$

This shows that $\mathrm{Stab}(V)$ is an algebraic subgroup of dimension $\leq \dim(V)$. We remark that equality of the dimensions holds if and only if V is a translate of a torus.

Let l be a positive integer. We are interested in relations between the degree of V and the degrees of $[l]^{-1}V = \{\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n \text{ s.t. } \boldsymbol{\alpha}^l \in V\}$ and of $[l]V = \{\boldsymbol{\alpha}^l \text{ s.t. } \boldsymbol{\alpha} \in V\}$.

Proposition VI.3.1 *We have*

$$\deg([l]^{-1}V) = l^{\mathrm{codim}(V)} \deg(V)$$

and

$$\deg([l]V) = \frac{l^{\dim(V)} \deg(V)}{|\mathrm{Ker}[l] \cap \mathrm{Stab}(V)|}. \tag{VI.3.9}$$

Proof. This is a special case of a general result of [24]. We give a sketch of the proof. Let us prove the first formula. For a hypersurface, this statement is clear. Indeed, let f be an equation of V. Then $f(\mathbf{x}^l)$ is an equation of $[l]^{-1}V$. We consider the general case. Let d be the dimension of V and let $W_1, \ldots, W_d$ be generic hypersurfaces of degree $D_1, \ldots, D_d$ such that $X = V \cap W_1 \cap \cdots \cap W_d$ is a finite set of $\deg(V) D_1 \cdots D_d$ points. Then $[l]^{-1}X = [l]^{-1}V \cap [l]^{-1}W_1 \cap \cdots \cap [l]^{-1}W_d$ is a set of cardinality $l^n|X|$. On the other hand, for what we have seen for hypersurfaces, this set has cardinality $\deg([l]^{-1}V) l^d D_1 \cdots D_d$. Thus $\deg([l]^{-1}V) = l^{n-d} \deg(V)$ as required.

The equality (VI.3.9) follows from the previous one. Indeed $[l]^{-1}[l]V = \mathrm{Ker}[l]V$ and $\mathrm{Ker}[l]V$ is a union of

$$\frac{l^n}{|\mathrm{Ker}[l] \cap \mathrm{Stab}(V)|}$$

distinct components. Thus

$$l^{\mathrm{codim}(V)} \deg([l]V) = \deg([l]^{-1}[l]V) = \frac{l^n \deg([l]V)}{|\mathrm{Ker}[l] \cap \mathrm{Stab}(V)|}.$$

□

VI.3.1 Heights of subvarieties

Let $\boldsymbol{\alpha} = (\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}^n(K)$ and let K be any number field containing $\alpha_0, \ldots, \alpha_n$. We define the Weil height of $\boldsymbol{\alpha}$ by:

$$h(\boldsymbol{\alpha}) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log \max\{|\alpha_0|_v, \ldots, |\alpha_n|_v\}.$$

As for the height of algebraic numbers, this definition does not depend on the number field K; moreover, by the product formula, it does not depend on the projective coordinates of $\boldsymbol{\alpha}$.

This provides a height function $\hat{h}(x_1, \ldots, x_n) = h(1 : x_1 : \cdots : x_n)$ on $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$. The following properties hold:

i) the function $\hat{h}$ is a positive function on $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$, vanishing only on its torsion points;

ii) $\hat{h}(\boldsymbol{\alpha\beta}) \leq \hat{h}(\boldsymbol{\alpha}) + \hat{h}(\boldsymbol{\beta})$. Moreover, if $\boldsymbol{\zeta}$ is a torsion point, $\hat{h}(\boldsymbol{\zeta\alpha}) = \hat{h}(\boldsymbol{\alpha})$. If $n \in \mathbb{N}$ then $\hat{h}(\boldsymbol{\alpha}^n) = n\hat{h}(\boldsymbol{\alpha})$;

iii) a subset of $\mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$ of bounded height and bounded degree is finite (Northcott's theorem)

The proofs are similar to those in dimension 1.

On hypersurfaces we have a "natural" definition of height rising from an extension of the Mahler measure to polynomials in several variables. Let $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$; we define its Mahler measure as:

$$M(P) = \exp \int_0^1 \cdots \int_0^1 \log \left| P\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| dt_1 \ldots dt_n$$

and we make the convention $M(0) = 0$. As in dimension 1, the Mahler measure is a multiplicative function. Moreover, if $\varphi(\mathbf{x}) = (\mathbf{x}^{\boldsymbol{\lambda}_1}, \ldots, \mathbf{x}^{\boldsymbol{\lambda}_m})$ is a finite monoidal morphism, then $M(P(\mathbf{x})) = M(P(\mathbf{x}^{\boldsymbol{\lambda}}))$. Let K be a number field and let $f \in K[\mathbf{x}]$ be a polynomial. We define

$$\hat{h}(f) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log M_v(f),$$

where $M_v(f)$ is the maximum of the v-adic absolute values of the coefficients of f if v is non archimedean, and $M_v(f)$ is the Mahler measure of σf if v is an archimedean place associated with the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. As for the Weil height, this definition does not depend on the field K containing the coefficients of f and $\hat{h}$ defines a positive and additive function on $\overline{\mathbb{Q}}[\mathbf{x}]$. Let

$$V = \{\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n \text{ s.t. } f(\boldsymbol{\alpha}) = 0\}$$

be a hypersurface in $\mathbb{G}_{\mathrm{m}}^n$ defined by some square-free polynomial $f \in K[\mathbf{x}]$. We define the normalized height of V as

$$\hat{h}(V) = \hat{h}(f).$$

This definition does not depend on the equation we choose for V. Let $\varphi\colon \mathbb{G}_{\mathrm{m}}^n \to \mathbb{G}_{\mathrm{m}}^n$ be a finite monoidal morphism. We also remark that $\hat{h}(\varphi^{-1}(V)) = \hat{h}(V)$.

Following Schinzel, we say that an irreducible $f \in \mathbb{Z}[\mathbf{x}]$ is an *extended cyclotomic polynomial* if there exist a cyclotomic polynomial ϕ and $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \in \mathbb{Z}^n$ such that

$$f(\mathbf{x}) = \pm \mathbf{x}^{\boldsymbol{\lambda}} \phi(\mathbf{x}^{\boldsymbol{\mu}}) \, .$$

In other words, an irreducible polynomial $f \in \mathbb{Z}[\mathbf{x}]$ is extended cyclotomic if and only if the hypersurface $\{f = 0\}$ in $\mathbb{G}_{\mathrm{m}}^n$ is a union of torsion varieties. In this context, Zhang's theorem on the toric Bogomolov conjecture can be paraphrased as follows. Let $f \in \mathbb{Z}[\mathbf{x}]$ be irreducible. Then $M(f) = 1$ if and only if $f = \pm x_j$ or if f is an extended cyclotomic polynomial. This result was proved earlier in [14], [25] and [34] independently.

The normalized height of an irreducible hypersurface has a nice behaviour under the action of pull back and pull out by multiplication by $[l]$. Indeed

$$\hat{h}([l]^{-1}V) = \hat{h}(V)$$

and

$$\hat{h}([l]V) = \frac{l^n \hat{h}(V)}{|\mathrm{Ker}[l] \cap \mathrm{Stab}(V)|}.$$

The first equality is a special case of the invariance of $\hat{h}(V)$ under inverse image by finite monoidal morphisms. The second equality follows from the first one and from the additivity of $\hat{h}$, exactly as the corresponding formulas for the degree.

The normalized height of a hypersurface can be computed as a limit. Let $f \in \mathbb{C}[\mathbf{x}]$. From inequality (VI.2.3) we deduce by induction on n (see [28] for details)

$$\|f\|_1 \leq 2^{d_1 + \cdots + d_n} M(f),$$

where $d_1, \ldots, d_n$ are the partial degrees of f. Let $\| \cdot \|$ be any norm on $\mathbb{C}[\mathbf{x}]$ such that

$$\log \|f\| = \log \|f\|_1 + O(\deg f) \tag{VI.3.10}$$

We define a height on hypersurfaces of $\mathbb{G}_{\mathrm{m}}^n$ by choosing the norm $\| \cdot \|$ at the archimedean places. Let as before

$$V = \{\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n \ \text{ s.t. } \ f(\boldsymbol{\alpha}) = 0\}$$

be a hypersurface in $\mathbb{G}_{\mathrm{m}}^n$ defined by some square-free polynomial $f \in K[\mathbf{x}]$. Let us define

$$h(V) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log H_v(f),$$

where $H_v(f) = M_v(f)$ if v is non archimedean, and $H_v(f) = \|\sigma f\|$ if v is an archimedean place associated with the embedding $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. Then,

$$\hat{h}(V) = h(V) + O(\deg(V)). \tag{VI.3.11}$$

Let l be a positive integer. Using the relations between degrees and heights of V and $[l]V$ we see that

$$\hat{h}(V) = \frac{\hat{h}([l]V) \deg(V)}{l \deg([l]V)}.$$

Thus, replacing in (VI.3.11) V by $[l]V$,

$$\hat{h}(V) = \frac{h([l]V) \deg(V)}{l \deg([l]V)} + O(l^{-1} \deg(V)).$$

This shows
$$\lim_{l\mapsto\infty} \frac{h([l]V)\deg(V)}{l\deg([l]V)} = \hat{h}(V).$$

The last formula suggests a "simple" definition of normalized height on subvarieties of $\mathbb{G}_{\mathrm{m}}^n$, due to Philippon [30]. We start by choosing a height on subvarieties. Let V be a d dimensional irreducible subvariety and let F be the Chow form of its Zariski closure in $\mathbb{P}^n$. The Chow form is an irreducible multihomogeneous polynomial $F(u_0^1,\ldots,u_n^1,\ldots,u_0^{d+1},\ldots,u_n^{d+1})$ vanishing precisely if the intersection of V with the linear space
$$\{\mathbf{x}\in\mathbb{P}^n \text{ s.t. } u_0^1x_0+\cdots+u_n^1x_n=\cdots=u_0^{d+1}x_0+\cdots+u_n^{d+1}x_n=0\}$$
is non empty. We define a height $h(V)$ as the height of the hypersurface in $\mathbb{G}_{\mathrm{m}}^{(d-1)n}$ defined by $\{F=0\}$, where one choose any reasonable norm at the archimedean places (*i.e.* a norm satisfying (VI.3.10)). David and Philippon (see [20]) prove that the limit
$$\hat{h}(V) = \lim_{l\to+\infty} \frac{h([l]V)\deg(V)}{l\deg([l]V)}$$
exists. We can see (compute the Chow form) that this definition of normalized height specializes to the previous ones if V is a point or if V is a hypersurface (see [20]). Moreover:

i) the function $\hat{h}(\cdot)$ is non-negative;

ii) for every $l\in\mathbb{N}$ we have
$$\hat{h}([l]^{-1}V) = l^{\mathrm{codim}(V)-1}\hat{h}(V)$$
and
$$\hat{h}([l]V) = \frac{l^{\dim(V)+1}\hat{h}(V)}{|\mathrm{Ker}[l]\cap\mathrm{Stab}(V)|}.$$

iii) for every torsion point $\boldsymbol{\zeta}$ we have $\hat{h}(\boldsymbol{\zeta}V)=\hat{h}(V)$.

For further details on the construction of the normalized height on tori and abelian varieties, see [30].

VI.3.2 Small height problems

Using properties ii) and iii) of the normalized height, we see that a torsion variety $V=\boldsymbol{\zeta}H$ has height zero. Indeed, if $\boldsymbol{\zeta}$ is a torsion point and H is a subtorus, then $\hat{h}(\boldsymbol{\zeta}H)=\hat{h}(H)$ and $\hat{h}(H)=\hat{h}([l]H)=l\hat{h}(H)$ for any $l\in\mathbb{N}$ (since $H=[l]H$ and $|\mathrm{Ker}[l]\cap H|=l^{\dim(H)}$).

Are torsion varieties the only varieties of zero height? The answer is positive; more precisely, this question is equivalent to the multiplicative analogue of the former Bogomolov's conjecture. Let us define, for $\theta>0$,
$$V(\theta)=\{\boldsymbol{\alpha}\in V \text{ s.t. } \hat{h}(\boldsymbol{\alpha})\le\theta\}$$
and let $\hat{\mu}^{\mathsf{ess}}(V)$ ("essential minimum") be the infimum of the set of $\theta>0$ such that $V(\theta)$ is Zariski dense in V. Theorem VI.1.2 asserts that $\hat{\mu}^{\mathsf{ess}}(V)=0$ if and only if V is torsion. By a special case of an inequality of Zhang (see [37], theorem 5.2.), we have, for an irreducible V,
$$\hat{\mu}^{\mathsf{ess}}(V) \le \frac{\hat{h}(V)}{\deg(V)} \le (\dim(V)+1)\hat{\mu}^{\mathsf{ess}}(V). \tag{VI.3.12}$$

This inequality shows that $\hat{h}(V)=0$ if and only if $\hat{\mu}^{\mathsf{ess}}(V)=0$. The problem of finding sharp lower bounds for $\hat{\mu}^{\mathsf{ess}}(V)$ for non-torsion subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ is a generalization of Lehmer's problem. Lower bounds for the essential minimum of a non-torsion subvariety will depend on some geometric invariants of V, for instance its degree. Moreover, if we do not make any further geometric assumption on the variety, such a bound must also depend on its field of definition ("arithmetic case"). Indeed, let H be a proper subtorus of $\mathbb{G}_{\mathrm{m}}^n$ and let $\boldsymbol{\alpha}_n$ be a sequence of non-torsion points whose height tends to zero (for instance, $\boldsymbol{\alpha}_n = (2^{1/n}, \ldots, 2^{1/n})$). Then, the varieties $V_n = H\boldsymbol{\alpha}_n$ have fixed degree $\deg(H)$ and essential minimum $\hat{\mu}^{\mathsf{ess}}(V_n) \leq \hat{h}(\boldsymbol{\alpha}_n) \to 0$. In spite of that, if we further assume that V is not a translate of a proper subtorus (even by a point of infinite order), then Bombieri and Zannier [16] proved that the essential minimum of V can be bounded from below only in terms of the degree of V ("geometric case").

As an exercice, we remark that this result of Bombieri and Zannier gives an alternative proof of Schinzel's result stated in section VI.2.2. Let L be a CM field and let $\alpha \in L^*$ such that $|\alpha| \neq 1$. We consider the curve $\mathcal{C} \subseteq \mathbb{G}_{\mathrm{m}}^2$ defined by the equation

$$f(x, y) = (x - \alpha)y - (\overline{\alpha}x - 1).$$

Since $\overline{\alpha} \neq \alpha^{-1}$ this curve is irreducible. Moreover, it is easy to see that $\mathcal{C}$ is not a translate of a subgroup. By the quoted result of Bombieri and Zannier, $\hat{h}(\mathcal{C}) \geq c > 0$ for some c which *does not depend* on α. Let v be an archimedean place associate with the embedding σ. Then

$$\log M_v(f) = \log M(x - \sigma\alpha) + \log M\left(y - \frac{\sigma(\overline{\alpha})x - 1}{x - \sigma\alpha}\right)$$

where we have extended $M(\cdot)$ to $\mathbb{C}(x, y)$ by multiplicativity. By (VI.2.1), $\log M(x-\sigma\alpha) = \log^+ |\sigma\alpha|$ and

$$\log M\left(y - \frac{\sigma(\overline{\alpha})x - 1}{x - \sigma\alpha}\right) = \int_0^1 \log^+ \left|\frac{\sigma(\overline{\alpha})e^{2\pi it} - 1}{e^{2\pi it} - \sigma\alpha}\right| dt.$$

This last quantity is zero. Indeed $\sigma(\overline{\alpha}) = \overline{\sigma(\alpha)}$ (recall that L is CM) and, for z, $\beta \in \mathbb{C}$ with $|z| = 1$,

$$\left|\frac{\overline{\beta}z - 1}{z - \beta}\right| = 1.$$

Thus $\log M_v(f) = \log^+ |\alpha|$ for $v \mid \infty$. Let now $v \nmid \infty$. An easy computation shows that

$$M_v(f) = \max(|\alpha|_v, |\overline{\alpha}|_v, 1) \leq \max(|\alpha|_v, 1)\max(|\overline{\alpha}|_v, 1).$$

Putting all together we get $0 < c \leq \hat{h}(\mathcal{C}) \leq 2h(\alpha)$.

Let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^n$. We define the "absolute obstruction index" $\omega(V)$ of V as the minimum of $\deg(Z)$ where Z is a hypersurface containing V. Similarly, we define the "rational obstruction index" $\omega_{\mathbb{Q}}(V)$ as the minimum of $\deg(Z)$ where Z is a hypersurface defined over $\mathbb{Q}$ containing V. For instance, let α be an algebraic number of degree d. Then $\omega_{\mathbb{Q}}(\alpha) = d$. More generally, let $\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$. Then, by standard linear algebra,

$$\omega_{\mathbb{Q}}(\boldsymbol{\alpha}) \leq n[\mathbb{Q}(\boldsymbol{\alpha}) : \mathbb{Q}]^{1/n} \tag{VI.3.13}$$

Even more generally, let V be a subvariety of $\mathbb{G}_n^m$. Then, if V is irreducible,

$$\omega(V) \leq n\deg(V)^{1/\mathrm{codim}(V)}.$$

Similarly, if V is defined and irreducible over the rational field, $\omega_{\mathbb{Q}}(V) \leq n\deg(V)^{1/\mathrm{codim}(V)}$. Both inequalities are special cases of a result of Chardin [18].

It turns out that $\omega_{\mathbb{Q}}(V)$, and not the degree of V, is the right invariant to formulate the sharpest conjectures on $\hat{\mu}^{\mathsf{ess}}(V)$ in the "arithmetic case". Similarly, $\omega(V)$ is the right invariant in the "geometric case". Although, in order to get statements depending on ω we need to assume, in the geometric case, not only that V is not a translate but also that V is not contained in any proper translate. Indeed, consider a curve $\mathcal{C} \subseteq \mathbb{G}_{\mathrm{m}}^{n-1}$. Let $\mathcal{C}' = \mathcal{C} \times \{1\} \subseteq \mathbb{G}_{\mathrm{m}}^n$ and choose, for $l \in \mathbb{N}$, an irreducible component V_l of $[l]^{-1}\mathcal{C}'$. Then $\hat{\mu}^{\mathsf{ess}}(V_l) \mapsto 0$, while $\omega(V_l) = 1$ since V_l is contained in the hypersurface $x_n = 1$. We shall say that an irreducible variety V is "transverse" if it is not contained in any proper translate. Similarly, in the arithmetic case we need to assume that V is not in a torsion variety. Such a V will be called a "weak-transverse" variety. Let $\boldsymbol{\alpha} \in \mathbb{G}_{\mathrm{m}}^n(\overline{\mathbb{Q}})$. We remark that the 0-dimensional variety $\{\boldsymbol{\alpha}\}$ is weak-transverse if and only if $\alpha_1, \ldots, \alpha_n$ are multiplicatively dependent.

Let V be a weak-transverse subvariety of $\mathbb{G}_{\mathrm{m}}^n$. In [2] we conjecture that

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\mathbb{Q}}(V)} \tag{VI.3.14}$$

for some $c(n) > 0$. Observe that this conjecture generalizes Lehmer's one. In [2] (case $\dim V = 0$), [3] (case $\mathrm{codim} V = 1$) and [4] (general case) we prove the following analogue of Dobrowolski's theorem on $\mathbb{G}_{\mathrm{m}}^n$.

Theorem VI.3.2 *Let V be a weak-tranverse subvariety of $\mathbb{G}_{\mathrm{m}}^n$ of codimension k. Let us assume that V is not contained in any torsion variety. Then there exist two positive constants $c(n)$ and $\kappa(k) = (k+1)(k+1)!^k - k$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega(V)} \left(\log 3\omega_{\mathbb{Q}}(V)\right)^{-\kappa(k)}.$$

Sometimes this theorem produces lower bounds for the height of algebraic numbers which are even sharper than what is expected by Lehmer's conjecture. Let $\alpha_1, \ldots, \alpha_n$ multiplicatively independent algebraic numbers of height $\leq h$, lying in a number field of degree d. Then $\hat{\mu}^{\mathsf{ess}}(\boldsymbol{\alpha}) \leq h$ and, by (VI.3.13),

$$\omega_{\mathbb{Q}}(\boldsymbol{\alpha}) \leq n d^{1/n}.$$

Thus, by theorem VI.3.2,

$$h \geq \frac{c(n)}{d^{1/n}} \left(\log 3d\right)^{-\kappa(n)}.$$

for some $c(n) > 0$.

Assuming that the subvariety V is tranverse, we now look for lower bounds for $\hat{\mu}^{\mathsf{ess}}(V)$ which do not depend on the field of definition of V (geometric case). In [5] we conjecture that

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\overline{\mathbb{Q}}}(V)}.$$

for some $c(n) > 0$. In the same paper we prove:

Theorem VI.3.3 *Let V be a transverse subvariety of $\mathbb{G}_{\mathrm{m}}^n$ of codimension k. Then there exist two positive constants $c(n)$ and $\lambda(k) = \left(9(3k)^{(k+1)}\right)^k$ such that*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \frac{c(n)}{\omega_{\overline{\mathbb{Q}}}(V)} \left(\log 3\omega_{\overline{\mathbb{Q}}}(V)\right)^{-\lambda(k)}.$$

The proofs of theorems VI.3.2 and VI.3.3 require several technical tools. By contradiction, we assume in both proofs that the essential minimum is sufficiently small. We then start following the usual steps of a transcendence proof: interpolation (construction of an auxiliary function), extrapolation and zero estimates. Concerning the last step, in both cases these proofs become very technical. In diophantine analysis a classical zero lemma (as [29]) is normally enough to conclude the proof. On the contrary, in [2] we need a more complicated zero lemma. As a consequence, this forces to extrapolate over different sets of primes. In Dobrowolski's proof one construct, using Siegel's Lemma, an auxiliary function F which vanishes on α. Then we extrapolate by proving that F must also vanish at α^p at least for small primes p. In the proof of theorem VI.3.2 (in the

0 dimensional case which is the hardest one) we construct an auxiliary function vanishing on $\boldsymbol{\alpha}$ and then we extrapolate by proving that F must also vanish at $\boldsymbol{\alpha}^{p_1\cdots p_n}$ for p_j small primes. The zero lemma we alluded before shows that for some $l = p_1 \cdots p_n$ the obstruction index $\omega_{\mathbb{Q}}(\boldsymbol{\alpha}^l)$ is pathologically smaller than $\omega_{\mathbb{Q}}(\boldsymbol{\alpha})$. Unfortunately, it seems hard to find lower bound for $\omega_{\mathbb{Q}}(\boldsymbol{\alpha}^l)$ in terms of $\omega_{\mathbb{Q}}(\boldsymbol{\alpha})$. Thus, we cannot easily conclude easily the proof. To avoid this problem, we start again the whole construction replacing $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}^l$. To ensure that the process end at some moment, we need a cumbersome induction ("descent step").

The situation is quite similar in the original proof of theorem VI.3.3. We construct again an auxiliary function vanishing on V and then we extrapolate by proving that F must also vanish on $\ker[p_1 \cdots p_n]V$ for p_j small primes. We need again a variant of a zero lemma which use the fact that our set of translation (the union of $\ker[p_1 \cdots p_n]$) is actually a union of big subgroups. Using this new zero lemma we succeed to show that again for some $l = p_1 \cdots p_n$ the obstruction index $\omega_{\mathbb{Q}}([l]V)$ is pathologically small than $\omega_{\mathbb{Q}}(V)$. As in the arithmetic situation, we cannot conclude easily and we need again a cumbersome descent step.

In [10] we recently succeed to drastically simplify the proof of theorem VI.3.3. The new proof encodes the classical diophantine analysis in an inequality involving some parameters, the essential minimum of a subvariety of $\mathbb{G}_{\mathrm{m}}^n$ and two Hilbert's functions. To decode the diophantine information we use a sharp lower bound for the Hilbert function due to Chardin and Philippon [19]. Finally, a delicate reduction process allows us to obtain the desired result. Possibly, this new method also applies in the arithmetic case.

We now consider the problem of the description of small points. Let V be a non-torsion variety of $\mathbb{G}_{\mathrm{m}}^n$ and define

$$V^* = V \setminus \bigcup_{\substack{B \subseteq V \\ B \text{ torsion}}} B.$$

By the former Manin-Mumford conjecture, V^* is a Zariski open set, indeed $V \setminus V^*$ is a finite union of torsion varieties. As mentioned in the introduction, an equivalent version of theorem VI.1.2 says that the height on $V^*(\overline{\mathbb{Q}})$ is bounded from below by a positive quantity:

$$\hat{\mu}^*(V) = \inf_{\boldsymbol{\alpha} \in V^*} \hat{h}(\boldsymbol{\alpha}) > 0.$$

Remark that obviously $\hat{\mu}^*(V) \leq \hat{\mu}^{\mathsf{ess}}(V)$. Hence one could hope, in analogy to (VI.3.14), that

$$\hat{\mu}^*(V) \geq \frac{c(n)}{\omega_{\mathbb{Q}}(V)}$$

for some constant $c(n) > 0$. This lower bound is not always true, as the following example shows.

Let α_k be a sequence of algebraic numbers whose height is positive and tends to zero as $k \to +\infty$. Let us consider

$$V_k = \{(\alpha_k, x_2, x_3) \in \mathbb{G}_{\mathrm{m}}^3 \text{ s.t. } \alpha_k^2 + \alpha_k^3 - x_2 - x_3 = 0\}.$$

The height of $\boldsymbol{\alpha}_k = (\alpha_k, \alpha_k^2, \alpha_k^3) \in V_k \setminus V_k^*$ tends to zero and $\omega_{\mathbb{Q}}(V) \leq 3$, since

$$V_k \subseteq \{x_1^2 + x_1^3 - x_2 - x_3 = 0\}.$$

In [6] we formulate the following conjecture. Let V be a non-torsion variety of $\mathbb{G}_{\mathrm{m}}^n$ defined by equation of degree $\leq \delta$ with integer coefficients. Then there exists a constant $c(n) > 0$ such that

$$\hat{\mu}^*(V) \geq \frac{c(n)}{\delta}.$$

In the same article, using a variant of theorem VI.3.2 and an additional induction, we prove this conjecture up to a logarithmic factor:

$$\hat{\mu}^*(V) \geq \frac{c(n)}{\delta}(\log 3\delta)^{-\kappa(n)}.$$

for some $c(n) > 0$ and where $k(n)$ is as in theorem VI.3.2.

We make a similar analysis in the geometric case. Let V be a tranverse subvariety of $\mathbb{G}_{\mathrm{m}}^n$ and define, as in [16],

$$V^0 = V \setminus \bigcup_{B \subseteq V} B.$$

where the union is now on the set of translates B of tori of dimension 1. Again $V \setminus V^0$ is an open set (see [16] and [33]); Bombieri and Zannier prove that, outside a finite set, the height on V^0 is bounded from below by a positive quantity depending on the degree of V (and not on its field of definition). More precisely, assume that V is defined by equation of degree $\leq \delta$. Schmidt [33] proves that the set of points $\boldsymbol{\alpha} \in V^0$ such that $\hat{h}(\boldsymbol{\alpha}) < q^{-1}$ is finite, of cardinality $\leq q$, where

$$q = \exp\left((4n)^{2\delta(2n)^\delta}\right).$$

David and Philippon [20] improve this result. They show that the above assertion still hold choosing:

$$q = n\Big(2^{n+4d+22} D (\log(D+1))^{2/3}\Big)^{7^d},$$

where D is the degree of the Zariski closure of V in $(\mathbb{P}^1)^n \subseteq \mathbb{P}^{2n-1}$ and d is the dimension of V.

Using a variant of theorem VI.3.3 and an additional induction, in [7] we prove that, for all but finitely many $\boldsymbol{\alpha} \in V^0(\overline{\mathbb{Q}})$,

$$\hat{h}(\boldsymbol{\alpha}) \geq c(n)\delta^{-1} \left(\log(3\delta)\right)^{-\lambda(n-1)},$$

where $c(n) > 0$ and $\lambda(k) = \left(9(3k)^{(k+1)}\right)^k$.

The proof of [7] gives no information on the cardinality of the set of points of pathologically small height. The new method introduced in [10] leads us to a complete quantitative description of the small points of a variety V. As a corollary of the main result of [10] we have:

Theorem VI.3.4 *Let $V \subseteq \mathbb{G}_{\mathrm{m}}^n$ be a (not necessarily irreducible) variety of dimension d defined by equation of degree $\leq \delta$. Let*

$$\theta = \delta\Big(200 n^5 \log(n^2\delta)\Big)^{dn(n-1)}.$$

Then,

$$|V^0(\theta^{-1})| \leq \theta^n.$$

Results of this shape have several applications. Using, among other deep ingredients, Schmidt's bound for the number of small points in V^0, Evertse, Schlickewei and Schmidt [23] prove an uniform bound for the number of arithmetic progression in the Skolem-Mahler-Lech theorem (theorem IV.4.7). They show that the set of zeros of a simple linear recurrence sequence in $\overline{\mathbb{Q}}$ of order $n \geq 1$ is the union of at most $\exp\left((6n)^{3n}\right)$ arithmetic progressions. For this kind of application V is a linear variety. Thus $\delta = 1$ and the important dependance is on n. Using theorem VI.3.4 instead of Schmidt's bound we can replace $\exp\left((6n)^{3n}\right)$ by $(8n)^{4n^5}$ in the result of Evertse, Schlickewei and Schmidt.

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